taking in the Power next above the greateſt given Power; then the other Part of the Multiplier being 1, its Product is the given Series. But this is to be ſubtracted from the former, (becauſe the Multiplier is $a-1$); and all the Terms of the two Series of Products being the ſame, except the greateſt of the firſt Series, and the leaſt of the other, it's manifeſt that the Difference, *i. e.* the Product ſought, is $a^{n+1}-a$, that is, $\overline{a+a^2+a^3, \&c.\ a^n} \times \overline{a-1} = a^{n+1}-a$. Hence dividing both Sides by $a-1$, it is $a+a^2+a^3, \&c.\ a^n = \frac{a^{n+1}-a}{a-1}$.

Whatever Power the Series begins at, the Reaſon of the Rule is the ſame; for the Products by a and by 1 will be the ſame Series, except the greateſt of the former, (which will be the next Power above the greateſt given Power,) and the leaſt of the other, (which is the leaſt given Power); ſo that the Difference of the two Series muſt be the Difference of theſe two.

THEOREM VIII.

THE Difference betwixt any two Powers of the ſame Root, is the Product of the Difference of the Root and 1, multiplied by the Sum of all the Powers of that Root from the leſſer given Power to that next below the greater.

Example 1. $a^n - a^r = \overline{a-1} \times \overline{a^r + a^{r+1} +, \&c. + a^{n-1}}$.

2. $a^n - a = \overline{a-1} \times \overline{a + a^2 + a^3, \&c.\ a^{n-1}}$.

DEMON. This is a manifeſt Conſequence of the preceding.

THEOREM IX.

TAKE the Series of Powers of any two Numbers (or Roots) to any the ſame Length or Index. To each of theſe Series prefix 1; then ſet the one of theſe Series under the other in a reverſe Order, and multiply the correſponding Terms of the one Series into thoſe of the other; then take the Sum of the Products: I ſay, if this Sum is multiplied by the Difference of the given Roots, the Product is equal to the Difference of their ſimilar Powers of the degree next above the higheſt in the Series.

$$\begin{array}{ccccc} a^4, & a^3, & a^2, & a, & 1. \\ 1, & b, & b^2, & b^3, & b^4. \\ \hline a^4 + & ba^3 + & b^2a^2 + & ab^3 + & b^4. \\ & & a-b. & & \\ \hline & & a^5 - b^5 & \text{Product.} & \end{array}$$

Example. Take any two Roots a, b; take their Powers to the 4th; the two Series formed and multiplied as in the Margin make the Series of Products, $a^4 + ba^3 + a^2b^2 + ab^3 + b^4$; which multiplied by $a-b$ produces $a^5 - b^5$.

Univerſally:

$$\begin{array}{cccccc} a^n, & a^{n-1}, & a^{n-2}, & \&c.\ a^2, & a, & 1. \\ 1, & b, & b^2, & \&c.\ b^{n-2}, & b^{n-1}, & b^n. \\ \hline a^n + & ba^{n-1} + & b^2a^{n-2}, & \&c\ a^2b^{n-2} + & ab^{n-1} + & b^n. \\ & & a-b. & & & \\ \hline & & a^{n+1} - b^{n+1}. & & & \end{array}$$

DEMON. The Reaſon of this appears the ſame way as that of *Theor*. 7. for $a^n \times a = a^{n+1}$, and $b^n \times b = b^{n+1}$; then the Product of a into every Term after a^n, is deſtroyed by that of b (which is to be ſubtracted) into the preceding.

COROL. Hence the Difference of any two Roots is an aliquot Part of the Difference of any their ſimilar Powers.

SCHOL. From the Doctrine of the next Book, *Chap*. 3. you'll find the Inveſtigation, and another Demonſtration of this and *Theor*. 7, and 8. *viz*. from the Conſideration of Geometrical Progreſſions. But I have placed them here, becauſe they have a Demonſtration independent of theſe Progreſſions: And *Theor*. 7. furniſhes us another Demonſtration for the ſumming of theſe Progreſſions.

ARITH-

ARITHMETICK.

BOOK IV.

The Doctrine of PROPORTION.

CHAP. I.

Explaining the general Nature of Proportion.

DEFINITIONS.

I. NUMBERS are compared in order to discover certain Relations they have to one another; and as to every Comparison there must be two Terms or Things, *viz.* one which is compared, and another to which it is compared, so must it be also in Numbers; where more particularly the Number compared is called the *Antecedent*, and the Number to which it is compared is called the *Consequent*. For Example; if we compare 3 to 4, 3 is called the *Antecedent* and 4 the *Consequent*.

SCHOL. Every Comparison is reciprocal, and includes a Comparison of each Term to the other; but because the Relation may be different according as the one or the other is made the Antecedent (as we shall see below); and because all Comparisons must be of things of like Species, therefore, in order to compare again the Relations of different Couplets of Numbers (as you'll find afterwards), we must carefully distinguish the two particular Comparisons that may be made in every Couplet; which is done by this Distinction of Antecedent and Consequent, according as they are applied to the two Numbers.

II. The Comparisons and consequent Relations of Numbers are of two kinds, distinguished by the Names of Arithmetical and Geometrical.

(1.) *Of* Arithmetical *Relation.*

If we compare Numbers so as to consider their simple Differences, or how much the Antecedent is greater or lesser than the Consequent; their Relation in that View is called *Arithmetical*; and the Difference of these two Numbers is called the *Arithmetical Ratio*, or *Exponent* of the Arithmetical Relation of the Antecedent to the Consequent. Example;

If

If we compare 3 to 5, the Arithmetical Ratio is 2, signifying this Relation, *viz.* that 3 is less than 5 by 2; and reciprocally, if we compare 5 to 3 the Ratio is also 2, signifying, in this Case, that the Antecedent 5 is greater than the Consequent 3, by 2.

If the Numbers compared are equal, their Arithmetical Ratio is 0, signifying their Equality.

Here then we see Arithmetical Relation distinguishable into two Kinds, *viz.* a Relation of Equality and Inequality; and the last again distinguishable into two Species, *viz.* a Relation of *Excess*, when the Antecedent is greatest, and of *Defect*, when the Antecedent is least. Now it would seem reasonable that the Exponents of different Relations, as the mutual Relations of unequal Numbers are, should be different; yet here it is the same Number, whether the Antecedent is greatest or least; and therefore that it may determine the Species of the Relation, we must apply the Words *Excess* and *Defect*; or some Mark to signify it in Writing. So if you say that 3 is the Arithmetical Ratio betwixt two Numbers, as 4 and 7, it may be applied two ways, according as we suppose the Antecedent greatest or least, for there are necessarily two mutual Relations betwixt two Things; yet by saying 3 in Excess or Defect, or using some Mark of Distinction in Writing, the Comparison is determined. But the Terms compared being known, the most simple and natural Method of determining the Comparison is to put the Antecedent always before the Consequent (as these Names do import), with the Word *To* betwixt them. For Example; To say the Relation of 4 *to* 7, is a particular and determinate Comparison, whereas to say the Relation *betwixt* 4 and 7 is ambiguous, for this may be either the Relation of 4 to 7, or of 7 to 4. When the Relation of different Couplets are to be again compared, this Determination is absolutely necessary.

(2.) *Of* Geometrical *Relation.*

If we compare two Numbers so as to consider how often the Antecedent contains or is contain'd in the Consequent, the Relation in that View is called *Geometrical*, and the Quote of the greater Term divided by the lesser, (which shews the *how oft* required) is called the *Geometrical Ratio*, or Exponent of the Geometrical Relation of the Antecedent to the Consequent. *Ex. gr.* If we compare 4 to 12, the Geometrical Ratio is 3, signifying this Relation of 4 to 12, *viz.* that it is contain'd in it 3 times; and reciprocally, the Ratio of 12 to 4, is 3, signifying the Relation of 12 containing 4, 3 times. Again; the Ratio of 4 to 19, or 19 to 4, is $4\frac{3}{4}$, signifying this Relation, *viz.* of 4 being contain'd in 19, or 19 containing 4 four times, and $\frac{3}{4}$ Parts of it. Of two equal Numbers, the Geometrical Ratio is always 1, expressing a Ratio of Equality.

Geometrical Relation is also distinguished into two kinds, one of Equality and another of Inequality; and the last again into two Species, *viz.* a Relation of *containing*, when the Antecedent is greatest, and of *being contained*, when the Antecedent is least; both which having the same Number for their Exponent, it must be applied differently according to the different Cases. But the Comparison is clearly and certainly determined in the same manner as is already explain'd in Arithmetical Relation, *viz.* by putting the Antecedent before the Consequent, with the Word *To* betwixt them; thus, to say the Relation of 4 to 12, is determinate and certain; and in this Case the Ratio 3 signifies that 4 is contain'd 3 times in 12: And if you call it the Ratio of 12 to 4, it signifies a Relation of *Containing*.

But again *observe*; There is another way of conceiving the Geometrical Relation of a lesser Number to a greater, whereby the Exponents of the two reciprocal Relations will be different: Thus, in comparing a lesser to a greater Number, we may consider what Part or Parts it is of the greater; and then the Fraction expressing this is the Geometrical Ratio: For Example; the Ratio of 4 to 12 is $\frac{1}{3}$; and so by this Method the Ratios, or Exponents of the Reciprocal different Relations of two Numbers, will always be different, and of themselves determine the Quality or Species of the Relation which they express, *i. e.* whether

2

whether the Antecedent is greater or leſſer, and that whether the Numbers compared are known or not: Thus, the Antecedent being greater, the Ratio will always be a whole or mixt Number; ſo the Ratio of 12 to 4 is 3, or of 14 to 4 is $3\frac{1}{2}$; but the Antecedent being leaſt, it will be a proper Fraction; ſo the Ratio of 4 to 12 is $\frac{1}{3}$, and of 4 to 14 is $\frac{2}{7}$. Now by the common Rules we find what Fraction the Antecedent is of the Conſequent, by dividing that Term by this; thus, if they are both Integers, the Quote and Fraction ſought is got by making the Antecedent Numerator, and the Conſequent Denominator, (which is perhaps not in its loweſt Terms, but that is no matter;) ſo 4 is $\frac{4}{14}$, or $\frac{2}{7}$ of 14. And if they are not both Integers, then we proceed by the Rules given in Diviſion of Fractions; ſo the Ratio of 4 to $13\frac{2}{3}$ is $\frac{12}{41}$, for $13\frac{2}{3}=\frac{41}{3}$, by which dividing 4, the Quote is $\frac{12}{41}$, which expreſſes what Fraction 4 is of $13\frac{2}{3}$ from the Nature of Diviſion.

Again; Since the Fraction expreſſing the Relation of a leſſer to a greater, is equal to the Quote of the Antecedent divided by the Conſequent; therefore, according to the Diſtinction of the reciprocal Relations of two Numbers, now explain'd, the Ratio is in all Caſes the Quote of the Antecedent divided by the Conſequent, expreſſing how oft the Antecedent, when it is the greater Term, contains the Conſequent; or, when it is leaſt, what Part or Parts, *i. e.* what Fraction it is of the Conſequent. And this was the Method of the *Antients* in explaining Geometrical Relation: But this we may reduce to a more uniform Notion, whereby the ſame general Definition comprehends both the mutual Relations of Inequality, and the Ratios they have to the ſame Number, thus: All Quotes are Fractions, and may be expreſſed fractionally; for by making the Dividend Numerator, and the Diviſor Denominator, when they are both Whole Numbers, that is a Fraction equivalent to the Quote; and if they are not both Integers, yet being divided by the common Rule, the Quote will come out in Form of a Fraction (tho' in ſome Caſes it may be equal to a Whole Number). And hence the Relation of a greater to a leſſer may be conceived under a like View with that of a leſſer to a greater, above explained, *viz.* as being equal to a certain Number of Parts of the Conſequent; and the improper Fraction expreſſing this is the Ratio, found out the ſame way as in the other Caſe. For Example; The Ratio of 7 to 5 is $\frac{7}{5}$, and of 7 to $2\frac{2}{5}$ it is $\frac{35}{12}$, for $2\frac{2}{5}=\frac{12}{5}$: From which we may deduce this general Definition of a Geometrical Relation, *viz.* That it is *the Relation of the Antecedent being equal to a certain Part or Parts,* i. e. *a certain Fraction proper or improper of the Conſequent*; which Fraction is the Ratio, expreſſing the Relation of a leſſer to a greater when it is a *proper* Fraction, but of a greater to a leſſer when it is improper. Hence, laſtly, we may reaſon about theſe Ratios under the Notion of Quotes or Fractions indifferently, theſe being in effect the ſame: And the great uſe of this is, that from the Doctrine of Fractions we can eaſily deduce the Theory of Geometrical Relation and Proportion, as you'll find afterwards done.

Now as to theſe different Ways of conceiving Geometrical Relation, it is indifferent as to the Truth of the Science built upon it, which we chuſe; for they anſwer equally true to all the Ends and Purpoſes of comparing the Relations of Numbers; becauſe they depend ſo upon one another, that in comparing the Relations in two different Couplets, if the Relations taken one Way are equal, they will be ſo taken the other way: Thus, if the Antecedent is greateſt, the Ratio is the ſame either way; for both the Methods of taking it coincide, and it is either a Whole or Mixt Number, or their Equivalent improper Fraction:

ction: If the Antecedent is least, then the Ratio taken the one way is the reciprocal Quote or Fraction of the Ratio taken the other way, and consequently two Ratios being equal the one way, they are so the other way also, because the Reciprocals of equal Fractions or Quotes are also equal. Thus, for example; If the Ratios of 10 to 14, and of 15 to 21, taken the one way, are both equal, *viz.* $1\frac{2}{5} = \frac{7}{5}$, then also must the Ratios taken the other way in both be $\frac{5}{7}$; or also thus, as $\frac{14}{10} = \frac{21}{15}$ is the Ratio the one way; so also will $\frac{10}{14} = \frac{15}{21}$ be the Ratio taken the other way (from *Lem.* 6. *Ch.* I. of Fractions.)

I have explain'd both these Methods of conceiving and expressing the Geometrical Relation of Numbers, because you'll meet with both in other Books on this Subject; but chiefly because I find it convenient to follow sometimes the one Method and sometimes the other; and sometimes for Variety and Illustration to apply both Methods.

COROLLARIES.

1. Comparing these different Notions of a Geometrical Ratio, with the common Operations of multiplying and dividing, this will evidently follow, *viz.* That the Ratio of two Numbers taken the first way, (which is always a whole or mixt Number) is that Number by which the lesser Term being multiplied, or the greater divided, (which soever of them is the Antecedent) gives the other Term; and taken the second way, (which may be either a whole or mixt Number, or a proper Fraction) it is that Number by which the Antecedent being divided, or the Consequent multiplied, (which soever of them is the greatest) gives the other: Or also thus; By whose Reciprocal the Antecedent being multiplied, or the Consequent divided, gives the other. If this is not evident, it may be made so thus:

If the Ratio is taken the first way, the Reason of this is manifest from the mutual Proof of Multiplication and Division. Thus: If A is less than B, and if $B \div A = d$, which is the Ratio; then $Ad = B$, and $A d - d$, that is, $B \div d = A$. Again; If the Ratio is taken the other way, suppose A is the Antecedent, the Ratio of A to B is this Quote $\frac{A}{B}$, and its reciprocal $\frac{B}{A}$. Now if A, B are both Integers, $\frac{A}{B}$ and $\frac{B}{A}$ are really fractional Expressions in Terms; and from the Rules of Fractions, it is plain that if the Antecedent A is divided by $\frac{A}{B}$, the Quote is B; or if B, the Consequent is multiplied by it, the Product is A. Again; Take the reciprocal Ratio $\frac{B}{A}$, and the Reverse of the former Operations produce the same Effect. Thus: A multiplied by $\frac{B}{A}$ produces B, and B divided by it quotes A: all which is evident from the common Rules. But again: If A, B are not both Integers, then suppose the Quote $\frac{A}{B}$, taken in its own proper Terms, is this fractional Expression $\frac{n}{o}$ (for every Quote is either a Fraction proper, or improper, which is expressible fractionally) then is $A = \frac{n}{o}$ of $B = B \times \frac{n}{o} = B - \frac{o}{n}$. Again; Since $A = \frac{n}{o}$ of B, therefore $B = \frac{o}{n}$ of $A = A \times \frac{o}{n} = A \div \frac{n}{o}$, wherein you see all the Parts of the Rule.

Example in Numbers. Thus: The Ratio of 12 and 4 taken the first way is 3; then $12 \div 3 = 4$, and $4 \times 3 = 12$. Again; Make 4 the Antecedent, and 7 the Consequent; the Ratio taken the second way is $\frac{4}{7}$; then $4 \div \frac{4}{7} = 4 \times \frac{7}{4} = 7$, and $7 \times \frac{4}{7} = 7 \div \frac{7}{4} = 4$.

2. *Hence*

2. *Hence again:* Having one Number given, and the Ratio betwixt it and another, we can easily find that other; providing also, that if the Ratio is a whole or mixt Number, it be determined whether the greater or lesser Term is sought. And if it is a proper Fraction, let it be determined whether the Number sought is the Antecedent or Consequent. *Exam.* Let there be given one Number 8, and 3 the Ratio betwixt it and another Number, to find that other Number. If 8 is the greater Term of the Relation, or if you call it the Antecedent, then $8 \div 3 = 2\frac{2}{3}$ is the Number sought; or if it is the lesser Term or Consequent, then is $8 \times 3 = 24$, the Term sought. Again; If 8 is the given Term, and $\frac{3}{4}$ the Ratio, then 8 being the Antecedent, $8 \div \frac{3}{4} = 8 \times \frac{4}{3} = \frac{32}{3} = 10\frac{2}{3}$ is the Consequent; but if 8 is the Consequent, then $8 \times \frac{3}{4} = 8 \div \frac{4}{3} = 6$ is the Antecedent.

Observe, To prevent such tedious Names as *Arithmetical* and *Geometrical Ratios,* we shall hereafter use the Word *Difference* for the first, and *Ratio* for the other.

III. *Of* PROPORTION.

When the same Kind of Relation, Geometrical or Arithmetical, is consider'd in each of two or more Couplets of Numbers, whose Terms are also similarly compared, *viz.* the lesser to the greater, or the greater to the lesser in each; then if these similar Relations are equal, *that is,* if their Exponents are equal Numbers, this Equality is call'd *Proportion*; which we may therefore more briefly define thus, *viz.* Proportion is the Equality of the *Differences* or *Ratios* of two or more Couplets of Numbers, whose Terms are similarly compared. And then the Numbers stated in Order as they are compared, the Antecedent of each Couplet before its Consequent are said to be proportional Numbers, or *Proportionals,* Arithmetical or Geometrical, according as we consider *Differences* or *Ratios.* *Exam.* 3, 4, and 5, 6 are arithmetically proportional; because their Antecedents are both least, and their Differences are equal, *viz.* 1; also these 7, 3, and 5, 9, the Antecedents being both greatest, and the Difference in each being 4.

Again; These 3, 6, and 4, 8 are geometrically proportional; the Antecedent in each being the lesser, and the Ratio equal, *viz.* 2 taken the one way, and $2\frac{1}{2}$ taken the other: Also these 9, 2, and 18, 4, whose Ratio taken either way is $4\frac{1}{2}$.

Universally: If $A - B = C - D$, then A, B, C, D are arithmetically proportional: and if $\frac{A}{B} = \frac{C}{D}$, then A, B, C, D, are geometrically proportional.

SCHOLIUMS.

1. Among Arithmetical Proportionals, 0 may be one Term. *Exam.* 0, 2, and 3, 5 have the same Difference, *viz.* 2. But all the Terms of Geometrical Proportionals must be real Quantities.

2. The two Numbers compared are ordinarily written with some Mark betwixt them, (the first written being the Antecedent, and the other the Consequent); and two Couplets, whose similar Relations are equal, (*i. e.* which are proportional) are written with some different Mark betwixt them: and it being convenient to distinguish the Marks of Arithmetical and Geometrical Relation, and their Equalities or Proportion, these distinguishing Marks I shall make thus: Betwixt two Numbers compared Arithmetically, (or in respect of Differences) I shall set a Point, thus, 3.5; and betwixt two such Couplets, whose Difference is equal, a Colon, thus, 3.5 : 4.6. Again; Betwixt two Numbers compared Geometrically, (or in respect of Ratios) set a Colon, thus, 3 : 4; and betwixt two such Couplets, whose similar Ratios are equal, a double Colon, thus, 3 : 4 :: 6 : 8; and then these Marks

ſave the Trouble of telling what kind of Compariſon is made of the Numbers. *Univerſally:* Theſe expreſs Arithmetical Proportionals, $a.b:c.d$; and theſe Geometrical, $a:b::c:d$; and then we read or expreſs the Proportionality thus, *viz.* In Arithmeticals we ſay, As much as the one Antecedent a exceeds, or wants of its Conſequent b; ſo much does the other Antecedent c exceed, or want of its Conſequent d; and when it is ſo betwixt four Numbers, they are Arithmetically proportional. In Geometricals we ſay, As oft as a contains, or is contain'd in b; ſo oft c contains, or is contained in d. Or alſo thus: What Fraction (proper or improper) a is of b; the ſame, or equal, Fraction is c of d.

Obſerve alſo, That Proportionality may be read thus more generally, *viz.* As a to b, ſo is c to d, Arithmetically or Geometrically; underſtanding this to ſignify what is above explain'd. And indeed the word Proportion, or Proportionality, is no more than a Word contriv'd to expreſs more briefly the Equality of Relation explain'd; ſo to ſay that four Numbers are proportional, is only ſaying all in one Word what muſt be ſaid in a great many, if explained at large, as has been now done.

The Diſtinction then betwixt Differences or Ratios, and Proportion, is this: A Difference or Ratio ariſes from the Compariſon of two or more ſimilar Relations, whoſe Exponents being equal make Proportion; which therefore can't exiſt without at leaſt four Terms (*i. e.* two Couplets); but obſerve, that the ſame Number may be antecedent in one Couplet, and conſequent in another; and therefore there may be Proportion, where there are but three different Numbers, if the one is twice taken: So theſe 3.4 5 are Arithmetically proportional, for $3.4:4.5$; and theſe Geometrically, $3:6:12$, for $3:6::6:12$.

It is to be obſerv'd too, that ſometimes the word Proportion is uſed for Ratio; ſo we ſay the Proportion of 3 to 6 for the Ratio of 3 to 6.

IV. *Of* Proportion, *Conjunct and Disjunct.*

When two or more Couplets are proportional, but ſo as they have no common Term (*i. e.* the Antecedent of none of them is the ſame Number as the Conſequent of another), they are called *Disjunct* Proportionals; as theſe Arithmeticals, $2.3:4.5:6.7$: and theſe Geometricals, $1:2::3:6::4:8$. But if the Conſequent of the firſt Couplet is the ſame Number as the Antecedent of the ſecond, and ſo on, the Conſequent of every Couplet the Antecedent of the next, theſe are called *Conjunct* (or continued) Proportionals; as theſe, $1.2:2.3:3.4$: and theſe, $1:2::2:4::4:8$. But in this Caſe the common Term need not be twice written; for it is enough to ſet all the different Terms in one continued Rank or Series, according to their Order of Compariſon; thus, $1.2.3.4$. or $1:2:4:8$; wherein it is underſtood that every Term is compared to the following; ſo that each is taken both as an Antecedent and a Conſequent, except the firſt, which is only Antecedent, and the laſt, which is only Conſequent. Such Series of Numbers are alſo called Arithmetical and Geometrical *Progreſſions.*

V. *Of* Proportion, *Direct and Reciprocal.*

When the Ratios of two Couplets are ſimilar and equal, the Proportion is called *Direct,* as in theſe, $1.2:3.4$: and theſe, $1:2::4:8$. But if they are diſſimilar, yet ſo as the one is equal to the Reciprocal of the other, the Proportion is called *Reciprocal*; as in theſe, $1.2:4.3$, and in theſe, $1:2::6:3$.

SCHOLIUM. This Diſtinction may be applied either to Arithmeticals or Geometricals, tho' it is commonly applied only to the laſt; as to which *obſerve,* That, according to *Definition* V. Proportion is no other thing than what is here called *Direct,* the other proceeding merely from the inverting one of the Couplets of a Direct or Proper Proportion: So that to ſay $1:2::6:3$ are proportional reciprocally, is no more than to ſay that theſe

these Numbers are proportional, if you invert any one of the Couplets, thus, 2 : 1 : : 6 : 3; this Distinction is therefore of no use in discovering or applying any Properties of Numbers; but it has taken rise from the Circumstances of some particular kind of mixt practical Questions; in which, tho' the Numbers are stated in the proper Order of Proportion, yet the Proportion is said to be reciprocal from a certain Consideration of the Order of the Terms as they lie in the Subject of the Question, of which you'll see Examples in the proper Place.

VI. *Of* Proportion, *Harmonical.*

If four Numbers are such, taken in a certain Order, that the first has the same Ratio to the fourth, as the Difference of the first and second has to the Difference of the third and fourth, these Numbers are said to be in *Harmonical* Proportion. Example: 9, 12, 16, 24, are harmonically proportional, *viz.* 9 : 24 : : 12 — 9 : 24 — 16, (or 9 : 24 : : 3 : 8) but if the first is to the fourth, as the Difference of the third and fourth to the Difference of the first and second, it is called *Contra-harmonical* Proportion; as 2, 8, 4, 6, where 2 : 6 : : 6 — 4 : 8 — 2 (*i. e.* 2 : 6 : : 2 : 6.).

SCHOLIUMS.

1. This kind of Proportion may exist also betwixt three Numbers, one of which is compared as a middle Term to both the Extremes; that is, if the first has the same Ratio to the third, as the Difference of the first and second to the Difference of the second and third; or as the Difference of the second and third to that of the first and second; for such coincide with the former by taking the middle Term twice. Example: 3, 4, 6, are Harmonical, because 3 : 6 : : 4 — 3 : 6 — 4 (*viz.* 3 : 6 : : 1 : 2); and these Contra-harmonical, 3, 5, 6; because 3 : 6 : : 6 — 5 : 5 — 3 (*viz.* 3 : 6 : : 1 : 2). And if in a Series of Numbers every three Terms are Harmonical, it is an Harmonical Series or Progression, (*i. e.* a Series of Conjunct or Continued Harmonicals.) *Example:* 10, 12, 15, 20, 30, 60, wherein every three Terms next together are Harmonical. And when of four Terms the two middle are different, that may be call'd Disjunct Harmonical Proportion.

2. But this considerable Difference is to be observ'd betwixt this kind of Proportion and the former, *viz.* That here in a Series of continued Harmonicals every four Terms adjacent will not also be Harmonical, as they are Geometrical or Arithmetical in these kinds of Progression. *Example:* Tho' 10, 12, 15, are Harmonical, and also 12, 15, 20; yet 10, 12, 15, 20, are not so.

3. If we take the Word *Proportion* strictly, according to the general Definition, for an Equality of similar Relations betwixt two Numbers, then it must be own'd there is no new Species of Proportion here; for there is no new kind of Relation in any two Terms, but only a mixt Comparison betwixt two Numbers and their Differences from others; and the Proportion constituted is really a Geometrical one, or an Equality of Ratios; not indeed of the given Numbers, but of the Ratio of the Extremes, compared to the Ratio of the Differences of these Extremes with the middle Terms: But tho' this is not strictly a Proportion among the given Numbers immediately compared together, yet it is certainly a new and Complex Relation (distinct from the immediate Arithmetical or Geometrical Relation betwixt the several Couplets of Numbers themselves), to which also the Name *Proportion* is applied.

This Denomination of *Harmonical* arises from *Musick*; because the Relations or Proportions of Sounds that make Harmony are found to be of the Kind here defin'd, of which you shall find a more particular Explication afterwards.

AXIOMS.

AXIOMS.

Observe, THE following *Axioms* are expressed so as to regard Geometrical Relations only; and the Ratio is to be understood as the Quote of the Antecedent divided by the Consequent: But by putting the Word *Difference* in place of Ratio, they are also applied to Arithmetical Relations or Differences.

Axiom I. Two or more equal Numbers (integral or fractional) A and B, have all the same Ratio to the same Number D, and D has the same Ratio to equal Numbers. Also these Numbers, A, B, *&c.* are equal, which have all the same Ratio to the same Number D, or to which the same Number D has the same Ratio. Again; Two equal Numbers have the same Ratio to any other two also equal; so if $A = B$, and $C = D$, then $A : C :: B : D$; this being the same as $A : C :: A : C$.

COROLL. If $A : B :: C : D$, and $L : M :: N : O$, then $\frac{A}{B} : \frac{L}{M} :: \frac{C}{D} : \frac{N}{O}$; for $\frac{A}{B} = \frac{C}{D}$, and $\frac{L}{M} = \frac{N}{O}$.

II. Two different Numbers, A and B, have different Ratios to the same Number D; and the greater or lesser of the two, A and B, has the greater or lesser Ratio to D. Again; The same Number D has a different Ratio to two different Numbers A and B, and it has reciprocally the lesser or greater Ratio to the greater or lesser of the two Numbers A, B. Also that is the greater or lesser of two Numbers A, B, which has the greater or lesser Ratio to one Number D; and that one of two Numbers, A, B, is the greater or lesser to which the same Number D has the lesser or greater Ratio.

COROLL. To the same three Numbers, A, B, C, there canont be two different Numbers that make a 4th Proportional; nor to two Numbers, two that are each a third Proportional.

III. If each of two Ratios is equal to a third, they are equal to one another. *Example:* If $A : B :: C : D$, and $A : B :: F : G$, then is $C : D :: F : G$.

COROLL. If one Ratio is equal to another, and this equal to a 3d, and this 3d to a 4th, and so on; the first Ratio is equal to the last, and each of them equal to each; so that the Terms of any two of them are proportional Numbers. *Example*: If $A : B :: C : D :: E : F :: G : H$, then is $A : B :: G : H$.

IV. Two equal Ratios, $A : B$ and $C : D$, are both equal to, or both greater, or both lesser than any third Ratio $G : H$. But *observe*, That the Reverse holds only when they are both equal to $G : H$, *i. e.* they are then only equal to one another; for they may be both greater or lesser, and yet not equal; there being Degrees of Inequality, but none of Equality.

V. Two equal Ratios are still equal in whatever Order they are taken: So that of two Couples that make Proportion, it is indifferent which of them is set first or last. *Example:* If $A : B :: C : D$, then also $C : D :: A : B$.

VI. All Ratios of Equality are equal Ratios; so $A : A :: B : B :: C : C$, *&c.* but equal Ratios are not always Ratios of Equality; and these things must be carefully distinguished: For a Ratio of Equality is the Ratio betwixt two equal Numbers; but two Ratio's may be equal, tho' they are both Ratio's of Inequality, *i. e.* of unequal Numbers.

General

General COROLLARIES.

I. *For* Arithmetical *Proportion.*

I. In *Addition*; If two Numbers are added together, then o, the two Numbers and the Sum are arithmetically proportional. *Example:* $2+4=6$, and $0.2:4.6$. Universally, $0.a:b.a+b$.

II. In *Subtraction*; o, the Difference Subtractor and Subtrahend, are arithmetically proportional. *Example*: $12-9=3$, and $0.3:9.12$: Universally, $0.a-b:b.a$.

II. *For* Geometrical *Proportion.*

III. In *Multiplication*; 1 is to any one of the Factors in the same Ratio as the other Factor is to the Product, *i. e.* these four are geometrically proportional. *Example:* $3\times 4=12$, and $1:3::4:12$, or $1:4::3.12$: for the Product 1 contains the one Factor as oft as the other expresses or contains Unity. Universally, $1:a::b:ab$, or $1:b::a:ab$.

IV. In *Division*; The Divisor or Quote is to Unity as the Dividend is to the other, (*i e.* these four are geometrically proportional). Example: $18\div 3=6$, and $6:1::18:3$, or $3:1::18.6$. Universally, $A\div d=q$, then $d:1::A:q$, or $q:1::A:d$; for d is contain'd in A, q times, or q in A, d times; that is, as oft as q or d expresses or contains Unity.

V. The Numerator of a simple Fraction is to the Denominator, as the Fraction (or Quantity expressed by it) is to Unity. Example: $\frac{2}{3}:1::2:3$. Universally, $\frac{a}{n}:1::a:n$; for a contains $\frac{a}{n}$ Parts of n, and $\frac{a}{n}$ signifies $\frac{a}{n}$ Parts of 1.

Observe, This is but a particular Case of the last; for, call $\frac{a}{n}=q$, and then $q:1::a:n$. Here the Terms a, n, are always integral; but in the other the Dividend and Divisor, a, d, may be integral or fractional. Again; 1 is to any Fraction as the Denominator to the Numerator; $1:\frac{a}{b}::b:a$; for $1\div\frac{a}{b}=\frac{b}{a}$. Hence, lastly, 1 is a mean Proportional betwixt any Fraction and its Reciprocal; so $\frac{a}{b}:1::1:\frac{b}{a}$.

VI. Of two equal Fractions, their Numerators and Denominators are geometrically proportional. *Example:* $\frac{2}{3}=\frac{4}{6}$, and $2:3::4:6$; also $2:4::3:6$; for because $\frac{2}{3}=\frac{4}{6}$, therefore $\frac{2}{4}=\frac{3}{6}$, (*Lem. 6. Chap.* 1. *B.* II.) Hence $2:4::3:6$. *Universally*, if $\frac{a}{n}=\frac{r}{s}$, then $a:n::r:s$, and $a:r::n:s$; for equal Fractions being equal Quotes, or Ratios, their Terms are in Proportion; it being the same thing to say that $\frac{a}{n}=\frac{r}{s}$, as to say that these are proportional $a:n::r:s$; the Equality of these Quotes or Fractions constituting the Proportion.

VII. Two Fractions having a common Denominator, are proportional with their Numerators. *Example*: $\frac{2}{5}:\frac{3}{5}::2:3$; for the Quote of $\frac{2}{5}$ by $\frac{3}{5}$ is the Quote of 2 by 3, *viz.*

$\frac{2}{3}$.

$\frac{2}{3}$. *Universally*, $\frac{a}{n} : \frac{b}{n} :: a : b$. For $\frac{a}{n} \div \frac{b}{n} = \frac{a}{b}$, (*Cor.* 1. to the Rule for Division of Fractions.)

VIII. Two Fractions having a common Numerator, are proportional with their Denominators taken in a reverse Order. *Example*; $\frac{2}{5} : \frac{2}{7} :: 7 : 5$; or $\frac{a}{n} : \frac{a}{r} :: r : n$; for $\frac{a}{n} \div \frac{a}{r} = \frac{r}{n}$, or $r \div n$, by Division of Fractions; for by the general Rule the Quote is $\frac{ar}{an}$, which reduces to $\frac{r}{n}$.

Observe, That in this Case the Fractions are said to be reciprocally proportional with their Denominators; because the one Fraction is to the other, as the Denominator of the last to that of the first; and is one Example of a true and direct Proportion, said to be reciprocal, because of the reverse Order in taking the Denominators, with respect to the Order in which the Fractions are taken.

IX. Any Fraction is to another in the same Ratio, as the Product of the Numerator of the first by the Denominator of the other, is to the Product of the Denominator of the first by the Numerator of the other, (*i. e.* as the new Numerators, when the Fractions are reduced to a common Denominator.) *Example*; $\frac{2}{3} : \frac{5}{7} :: 14 : 15$; or $\frac{a}{n} : \frac{r}{s} :: as : nr$; because the Quote of these two Products, or new Numerators, is the Quote of the two Fractions by Division of Fractions; and the Quotes are the Ratios.

X. Unity is a Geometrical Mean betwixt any Number and its Reciprocal. *Example*: $3 : 1 :: 1 : \frac{1}{3}$, and $\frac{2}{3} : 1 :: 1 : \frac{3}{2}$. *Universally* : $A : 1 :: 1 : \frac{1}{A}$, and $\frac{A}{B} : 1 :: 1 : \frac{B}{A}$.

XI. A Fraction compounded of two Fractions, or the Product of two simple Fractions, is to any of them as the Numerator of the other is to its Denominator. Thus, $\frac{A}{B}$ of $\frac{C}{D}$, or $\frac{AC}{BD} : \frac{A}{B} :: C : D$, and $\frac{AC}{BD} : \frac{C}{D} :: A : B$. In Numbers $\frac{2}{3}$ of $\frac{5}{7}$, or $\frac{10}{21} : \frac{2}{3} :: 5 : 7$.

Hence any square Fraction is to its Root, as the Numerator to the Denominator of the Root. Thus $\frac{A^2}{B^2} : \frac{A}{B} :: A : B$. In Numbers, $\frac{4}{9} : \frac{2}{3} :: 2 : 3$.

XII. Two Fractions are as their alternate Fractions. Thus; $\frac{A}{B} : \frac{C}{D} :: \frac{A}{C} : \frac{B}{D}$, the common Ratio being $\frac{AD}{BC}$. In Numbers, $\frac{2}{3} : \frac{5}{7} :: \frac{2}{5} : \frac{3}{7}$.

XIII. Any two Numbers are in the same Ratio with their Reciprocals taken reciprocally. *Example*: $2 : 3 :: \frac{1}{3} : \frac{1}{2}$; and $\frac{2}{3} : \frac{4}{7} :: \frac{7}{4} : \frac{3}{2}$. *Universally* in Integers, $A : B :: \frac{1}{B} : \frac{1}{A}$; and in Fractions $\frac{A}{B} : \frac{C}{D} :: \frac{D}{C} : \frac{B}{A}$, the common Ratio being $\frac{AD}{BC}$.

And

And obſerve, That ſince the Quote of any Number divided by another, is reducible to a fractional Expreſſion; alſo ſince the Reciprocal of that Quote is the Quote of the ſame Numbers, changing the Diviſor into the Dividend and this into that, which we call the reciprocal Quote; therefore the Quotes of any two Couplets of Numbers are in the ſame Ratio as the reciprocal Quotes taken reciprocally; which we may expreſs generally thus: $A \div B : C \div D :: D \div C : B \div A$: for let $A \div B = \frac{a}{b}$, and $C \div D = \frac{o}{n}$; then is $B \div A = \frac{b}{a}$, and $D \div b = \frac{n}{o}$, (by what is ſhewn in Diviſion of Fractions) and by what is now ſhown $\frac{a}{b} : \frac{o}{n} :: \frac{n}{o} : \frac{b}{a}$.

XIV. Any Number whatever is to any of its Multiples or Fractions in the ſame Ratio as any other Number is to its like Multiple or Fraction. This is the immediate Conſequence of the Definition; for that Likeneſs proceeds from, or conſtitutes an equal Ratio, and may be repreſented in any of theſe Forms, *viz.* $A : An : B : Bn$; or $A : A \div n :: B : B \div n$, as has been ſhewn in Diviſion of Fractions, where n is the Ratio; and according as it repreſents a whole Number or a Fraction, ſo will An, and $A \div n$ repreſent a Multiple or Fraction of A. And if $\frac{n}{m}$ is a Fraction, proper or improper, then this Truth may alſo appear thus: $A : \frac{n}{m}$ of $A :: B : \frac{n}{m}$ of B: for $\frac{n}{m}$ of $A = A \div \frac{m}{n}$, and $\frac{n}{m}$ of $B = B \div \frac{m}{n}$, as has been ſhewn in Diviſion of Fractions; and ſo $\frac{m}{n}$ is the Ratio, by which the Antecedents being divided give the Conſequents.

XV. Any two Numbers whatſoever are in the ſame Ratio, or proportional with any their like Multiples or Fractions; *i. e.* as the Products or Quotes of theſe Numbers multiplied or divided by any the ſame Number. Thus: $A : B :: An : Bn$; or $A : B :: A \div n : B \div n$; which Proportionality follows from this, That the like Fractions or Multiples of any two Numbers whatſoever, are the ſame Fractions one of the other as theſe Numbers are, (*Cor.* 5, 6. *Lem.* 2. *Chap.* 1. *B.* II.) and like or equal Fractions conſtitute equal Ratios.

XVI. The Quotes of the ſame Number divided ſeparately by any two different Diviſors are in the reciprocal Ratio of the Diviſors. Thus: $A \div n : H \div m :: m : n$; for whatever kind of Numbers A, n, m repreſent, $A \div n$, $A \div m$ repreſent certain Fractions into which theſe Quotes will reſolve; and $n \div A$, $m \div A$ repreſent the Reciprocal of theſe Fractions. Hence $A \div n : A \div m :: m \div A : n \div A$, (*Cor.* 13.) and $m \div A : n \div A :: m : n$, (by the laſt); therefore $A \div n : A \div m :: m : n$, (*Ax.* 3.)

XVII. Any the like Fractions of any two Numbers, are to one another in the ſame Ratio as any other like Fractions of the ſame two Numbers. Thus: $\frac{a}{b}$ of $A : \frac{a}{b}$ of $B :: \frac{c}{d}$ of $A : \frac{c}{d}$ of B; becauſe the Ratio of each of theſe Antecedents to its Conſequent, is that of $A : B$, (*Cor.* 15.); therefore they are equal, (*Ax.* 3.)

XVIII. Any two different Fractions of the ſame Number are to one another in the ſame Ratio as the ſame two Fractions of any other Number. Thus, $\frac{a}{b}$ of $A : \frac{c}{d}$ of $A :: \frac{a}{b}$ of $B : \frac{c}{d}$ of B. The Reaſon of this is $\frac{a}{b}$ of $A = A \div \frac{b}{a}$, and $\frac{c}{d}$ of $A = A \div \frac{d}{c}$, (as has been ſhewn in Diviſion of Fractions) and $A \div \frac{b}{a} : A \div \frac{d}{c} :: \frac{d}{c} : \frac{b}{a}$ [*Cor.* 16.]. For the

the same Reasons, $\frac{a}{b}$ of $B = B \div \frac{b}{a}$, and $\frac{c}{d}$ of $B = B \div \frac{d}{c}$: also $B \div \frac{b}{a} : B \div \frac{d}{c} :: \frac{d}{c} : \frac{b}{a}$. Hence, (by *Ax.* 3.) $A \div \frac{b}{a} : A \div \frac{d}{c} :: B \div \frac{b}{a} : B \div \frac{d}{c}$; that is, $\frac{a}{b}$ of $A : \frac{c}{d}$ of $A :: \frac{a}{b}$ of $B : \frac{c}{d}$ of B.

XIX. If one Number A is divided by another B, and the Quote q be divided or multiplied by another n, this last Quote or Product is the same Number that will be found by taking the Quote or Product of A divided or multiplied by the same n, and dividing it by B. Thus, if $A \div B = q$, then $An \div B = qn$, and $\overline{A \div n} \div B = q \div n$. The Reason is easy: For we shall suppose that $An \div B = Q$, then are q and Q Like Fractions of A and An, because of the same Divisor B, and therefore $A : An :: q : Q$, [*Cor.* 15.] But $A : An :: q : qn$ [*Cor.* 14.], therefore $Q = qn$ (*Ax.* 2. *Cor.*): And again, supposing $\overline{A \div n} \div B = Q$, then $A : A \div n :: q : Q$; but $A : A \div n :: q : q \div n$; hence $Q = q \div n$.

XX. If one Number A is divided by another B, and the Quote q be divided or multiplied by another n, this last Quote or Product is the same that will be found by taking reversly the Product or Quote of B, multiplied or divided by the same n, and by it dividing A. Thus; if $A \div B = q$, then $A \div Bn = q \div n$; also $A \div \overline{B \div n} = qn$: For suppose $A \div Bn = Q$, then $q : Q :: Bn : B$ [*Cor.* 16.]; but $Bn \div n = B$, therefore $q : Q :: Bn : Bn \div n$; also $q : q \div n :: Bn : Bn \div n$ [*Cor.* 14.]. Hence $Q = q \div n$ (*Ax.* 2.).

XXI. From these two last we see again evidently, that if any Divisor and Dividend are equally multiplied or divided, the Quote made of these new Numbers is the same as that made of the Numbers multiplied or divided: Thus $A \div B = An \div Bn$.

General SCHOLIUM.

It has been frequently observed, that such an Expression as this, $\frac{A}{B}$, does not represent directly and immediately a Fraction in Terms, unless A and B do both represent Integers; yet $\frac{A}{B}$ may very well express the Quote of A divided by B, tho' it's but a general and indeterminate Expression thereof; for it is in effect no more than a Sign or Mark for these Words, *The Quote of* A *divided by* B, and expresses the Quote only in the same indeterminate manner as these Words do. But now from what has been shewn in *Coroll.* 19. and 20. we have learn'd this very remarkable thing, *viz.* How any Quote, tho' it's only expressed in this general Form, may be multiplied or divided; *i. e.* how another Expression of the same kind may be found from the given Terms, equal to the Product or Quote of the given Quote by any given Number: For which this is the Rule, *viz.* Multiply or divide the given Dividend, and apply to the Product or Quote the given Number, and you have the Product or Quote sought. Again, reciprocally, divide or multiply the given Divisor, and apply the Product or Quote to the given Dividend, and you have also the Product or Quote sought. Thus, for Example: $\frac{A}{B} \times n = \frac{An}{B} = \frac{A}{B \div n}$, and $\frac{A}{B} \div n = \frac{A \div n}{B} = \frac{A}{Bn}$, taking these as general Expressions of Quotes: For suppose $\frac{A}{B} = q$, it's shewn that $qn = An \div B$, (*Coroll.* 19.); $= A \div \overline{B \div n}$ (*Coroll.* 20.): Also that $q \div n = \overline{A \div n} \div B$ (*Coroll.* 19.) $= A \div Bn$ (*Coroll.* 20.); wherefore taking these other Forms which express the

2

the ſame things, it is $\frac{A}{B} \times n = \frac{An}{B} = \frac{A}{B \div n}$, and $\frac{A}{B} \div n = \frac{A \div n}{B} = \frac{A}{Bn}$. *Obſerve* alſo, That theſe Expreſſions are right, whether n repreſent an Integer or Fraction; ſince to multiply or divide by a Fraction is a Mixt Operation of Multiplication and Diviſion; ſo that in this Caſe An, $A \div n$, Bn, $B \div n$, do all repreſent the Effect of a Mixt Operation, whoſe Parts are both juſtly applied to the Diviſor or Dividend, to repreſent the Multiplying or Dividing the Quote or Fraction $\frac{A}{B}$.

Upon which Rule this is to be *obſerved,* That Quotes expreſſed in this general manner are multiplied and divided the ſame way as Fractions are, (which are Quotes of Integers, and therefore but a particular Caſe of this general one) doing the ſame with the Diviſor and Dividend as with the Denominator and Numerator of a Fraction.

Hence follows, That all the Rules of Operations in Fractions are truly applicable to Quotes expreſſed in this general Form; becauſe all the Reaſons of theſe Operations in Fractions have been demonſtrated from the Conſideration of the Equi-multiplication and Diviſion of the Fraction, by equi-multiplying or dividing its Numerator and Denominator; which is the whole Foundation of the Demonſtrations of the Rules for multiplying and dividing Fractions. Then as to the Addition and Subtraction of Fractions, the Demonſtration takes in alſo the Conſideration of *Lemma* 2. and 3. *Ch.* I. *Book* II. which are of the ſame Uſe in the Addition and Subtraction of Quotes; becauſe all Quotes are ſuch Fractions of the Dividend as the Reciprocal of the Diviſor expreſſes (as has been frequently mention'd), which I ſhall here very briefly apply. In the firſt Place, Two Quotes are reduced to a common Diviſor the ſame way as two Fractions to a common Denominator: thus $\frac{A}{B}$ and $\frac{C}{D}$ are reduced to theſe, $\frac{AD}{BD}$ and $\frac{BC}{BD}$; becauſe the Equi-multiplication of the Diviſor and Dividend makes an equal Quote (*Coroll.* 21.) and $\frac{AD}{BD} + \frac{BC}{BD} = \frac{AD+BC}{BD}$: For whatever the Diviſor BD is, the Quote is that Fraction of the Dividend expreſſed by the Reciprocal of the Diviſor; and the Sum of the Like Fractions of two Numbers is the Like Fraction of the Sum of theſe Numbers (*Lemma* 2. *Ch.* I. *B.* II.). Again; $\frac{AD}{BD} - \frac{BC}{BD} = \frac{AD-BC}{BD}$ (*Lemma* 3. *Ch.* I. *Book* II.)

We muſt alſo *obſerve,* That beſides theſe common Operations, all the other Properties of Fractions whoſe Reaſon depends upon Multiplication and Diviſion of their Terms, are the ſame way applicable to Quotes; particularly, all theſe relating to Geometrical Proportion of their Terms, explained in the preceding *Corollaries*: To which remember to join theſe three Properties of equal Fractions in *Lem.* 6. *Ch.* I. *Book* 2, *viz.* That theſe Fractions or Quotes being equal $\frac{A}{B} = \frac{C}{D}$, then, 1°, $AD = BC$; 2°. $\frac{A}{C} = \frac{B}{D}$; 3°. $\frac{B}{A} = \frac{D}{C}$.

Now the Uſe of all that has been explained concerning theſe general and indeterminate Expreſſions of Quotes, is in the Demonſtration of the following Theory of Proportion, eſpecially the Geometrical Kind; for to make a general and compleat Theory, we muſt conſider both Integers and Fractions; and where any Theorem is true univerſally, whether applied to Integers or Fractions, then to make the Demonſtration alſo univerſal, and yet as ſimple as poſſible, it's neceſſary to make any Letter, A, or B, repreſent any Number, Integer or Fraction, indifferently; and then all the Properties of proportional Numbers depending upon the Conſideration of their Ratios (which are Quotes; ſo the Ratio of A to B is

B is the Quote of A divided by B, expressed $\frac{A}{B}$, or $A \div B$), and the Demonstrations depending upon the Multiplication and Division of these Quotes; therefore, by shewing that the Operations of Fractions and their other Properties are the same in Ratios or Quotes thus expressed, we learn an universal Method of Demonstration (so far as the Multiplication and Division of the Ratios or Quotes are concerned) whether the Ratio in the given Terms is a real Fraction, as when A, B are both Integers, or if it's only looked upon as a more general Expression of a Quote, as when A, B, are both, or one of them, Fractions; and indeed we shall find that as equal Ratios or Quotes, and equal Fractions, are in effect the same thing; so the Properties of Fractions already explained, applied to Quotes, do contain all the Truths hereafter proposed concerning proportional Numbers, or they may be deduced from them.

It's true indeed that there are various Methods of demonstrating this Theory; one of which is independent of the things now explained concerning Quotes, and is perhaps the most simple and easy in many Cases; yet other Methods are beautiful, and deserve to be considered, as they serve to enlarge our Ideas and Knowledge: But besides, for different Propositions different Methods are convenient; and as different Persons may 'tis probable be pleased, some with one, some with another way of representing and demonstrating the same Truth, so some will be pleased with a Variety of Ways; therefore the Things I have explained concerning Quotes were necessary; and I have accordingly used Demonstrations which suppose the Knowledge of them: And as I have used different Methods for different things, so I have also used a Variety of Demonstrations for some particular Truths, where it could be done without being tedious. To conclude then, When you find in the following Theory any Quote thus represented, $\frac{A}{B}$ (or $A \div B$) and the Product or Quote of this by any other, as n, or $\frac{r}{n}$ (whatever Kind these represent), performed and expressed as if $\frac{A}{B}$ or $\frac{r}{n}$ were real Fractions; as thus, $\frac{A}{B} \times n = \frac{An}{B}, = \frac{A}{B \div n}$: Or $\frac{A}{B} \times \frac{r}{n} = \frac{Ar}{Bn}$. Also $\frac{A}{B} \div n = \frac{A \div n}{B} = \frac{A}{Bn}$; or $\frac{A}{B} \div \frac{r}{n} = \frac{An}{Br}$. Remember that all this is the Application of what has been here explained.

Of the several Kinds of Ratios, *as they were distinguished by the* Ancients.

Before I end this Chapter, I shall give you an Account of the several Kinds into which Ratios were distinguished by the Ancients; and which are yet upon occasion made use of by some.

I. *Ratios* are either,

1. *Multiple*, when the Antecedent contains the Consequent a certain Whole Number of times, without a Remainder; *i. e.* when it is a Multiple of it. *Exam.* 4 to 1, or 12 to 3.
2. *Super-Particular*, When the Antecedent contains the Consequent once, and 1 over; as 4 to 3, or 5 to 4.
3. *Super-Partient*, When the Antecedent contains the Consequent once, and a Remainder greater than 1; as 5 to 3.

Observe, That as the same Relation may be betwixt different Terms; so 5 : 3 :: 10 : 6. These two Definitions suppose the lowest Terms of the Ratio: (Which is by some called the *Exponent* of the Relation, tho' I apply this to the Quote or Fraction made of any two Terms in the same Relation, because they are all equal; leaving them to be distinguished

guished by the lowest Terms, and such as are not so); For otherwise two Numbers may appear to be a Super-partient Ratio, which are Super-particular; so 4 : 3 :: 8 : 6; but 4 to 3 is Super-particular, and consequently so must 8 to 6; and yet this appears Super-partient, unless we restrain the Definitions to the lowest Terms.

4. *Multiple Super-particular*; When the Antecedent contains the Consequent oftner than once with 1 over; as 5 : 2, or 13 : 4.

5. *Multiple Super-partient*; When the Antecedent contains the Consequent oftner than once, and a Remainder greater than 1; as 12 : 5, or 18 : 7.

Again; They distinguished the Ratios of the lesser to the greater, by prefixing the Word *Sub* instead of *Super* in the preceding Names; thus, *Sub-Multiple*, as 1 : 4; *Sub-particular*, as 3 : 4; *Sub-partient*, as 3 : 5, and so on.

Also, They had particular Names for the several Species or Sub-divisions of each Kind. So the Multiple are either double, triple, &c. the Super-particular were called *Sesqui-altera*, as 3 : 2; *Sesqui-tertia*, as 4 : 3, and so on; setting the Word *Sesqui* before the Name of the lesser Term. For the *Super-partient*, they put the Name of the Number by which the Antecedent exceeds the Consequent betwixt the Words *Super* and *Partient*, and the lesser Term of the Ratio last of all; thus, 5 : 3 is called *Superbis partiens tertias*. But these Names are useless and troublesome, since we can much easier and more plainly express the Ratio by its Terms.

II. I shall next shew you a certain regular Method and Order, in which all possible Examples of each of these Kinds of Ratios may be found, and in their lowest Terms.

1. For *Multiple* Ratios make 1 the lesser Term, and any other Number from 2 *in infinitum*, the greater.

2. For *Super-particular* Ratios; Make any Number greater than 1 the lesser Term, and the next greater Number make the greater Term; that is, compare every Term of the natural Series of Numbers from 2; as 2, 3, 4, 5, 6, &c. to the next.

3. For *Super-partient* Ratios; Take any Number above 2 for the lesser Term, and to it add any lesser Number which is not an aliquot Part of it, nor the Multiple of any aliquot Part of it greater than 1, that Sum is the greater Term: Thus you may find all the Super-partient Ratios which can have any assumed Number for the lesser Term, supposing you can find all the aliquot Parts of any Number; as you'll learn in *Book* V. *Chap.* 1. The *Reason* of this is, That if n is not an Aliquot of a, nor the Multiple of any aliquot Part greater than 1, then a, and $a+n$ cann't have a common Measure (but 1); for if they have, then because it measures a and $a+n$, it must also measure n; and consequently n is either an aliquot Part or Multiple of some aliquot of a, contrary to Supposition. Lastly, because n is less than a, yet greater than 1; therefore $a+n$ to a is *Super-partient*. *Example*: 7 (= 4 + 3) : 4.

4. The *Multiple*-Super-particular and Super-partient Ratios are easily found from these Rules.

III. I shall end this with another Observation upon the Dependence of the different Kinds of Ratios upon one another, and upon the Ratio of Equality.

1. Take any two equal Numbers, as 1 : 1, one of them compared to the Sum of both, is in double Ratio, or as 1 to 2. Again; the lesser of these compared to the Sum of both is triple Ratio; as 1 : 3. By going on so, you'll have all the Multiple Ratios.

2. Of all the Species of Multiple Ratios, compare the greater Term to the Sum of both, you have all the Species of Super-particulars: Thus, from 1 : 2 comes 2 : 3, from 1 : 3 comes 3 : 4, and so on.

3. Of all the Super-Particulars, compare the lesser to the Sum, you have all the double Super-particulars: So from 2 : 3 comes 2 : 5, from 3 : 4 comes 3 : 7. &c. Again; From the double Super-particulars come the Triple, as from 2 : 5 comes 2 : 7; and from the triple comes the Quadruple, and so on.

4. Of

4. Of all the Super-particular Ratios, compare the greater to the Sum, and they are Super-partient, and they are particularly of that Kind in which the Excess and the lesser Term are Super-particular; thus from 3 : 2 come 5 : 3, whose Exponent is $1\frac{2}{3}$; from 4 : 3 come 7 : 4, whose Exponent is $1\frac{3}{4}$. Or if you take these two Series, *viz.* 5, 7, 9, *&c.* still increasing by 2, and this, 3, 4, 5, *&c.* increasing by 1, compare their Terms in Order, and you have all these Super-partient Ratios, *viz.* $\frac{5}{3}.\frac{7}{4}.\frac{9}{5}.\frac{11}{6}.\frac{13}{7}.$ *&c.*

5. From the Super-partient Ratios proceed their Multiples the same way as from the Super-particulars.

The *Reasons* of all these things are obvious.

CHAP. II.

Of Arithmetical Proportion.

Observe: *To shorten writing, I mark the Words* Arithmetical Proportion, *or* Arithmetically Proportional, *by this Character,* :*l*; *and* Continued Arithmetical Proportion *by this* ∸*l.*

§. 1. *Containing the more general Doctrine common to both* Conjunct *and* Disjunct Proportion.

Prob. I. *Three Numbers given, to find a* 4th :*l.*

Example 1.

Given. Sought.

2 . 5 : 7 . 10.

Operation.

$5-2=3$. then $7+3=10$.

Example 2.

Given. Sought.

9 . 7 : 4 . 2.

Operation.

$9-7=2$. then $4-2=2$.

Rule. TAKE the Difference of the first and second, and either add it to the third, (as in *Exam.* 1.) or subtract it from it, (as in *Exam.* 2.) according as the Antecedent is less or greater than the Consequent, (*i. e.* the first Term than the second) the Sum or Difference is the fourth sought. So universally to these, *a. b. c*, a 4th :*l* is $b-a+c$, (if *b* is greatest) or $c-\overline{a-b}=c-a+b$.

The Reason is plainly contain'd in the Definition of :*l*.

Corol. Since 3 Numbers :*l* are in effect the same Case as 4 Numbers, by taking the middle Term twice; therefore a 3d :*l* to two given Numbers is found by taking the Difference of the first and second, and adding it to, or subtracting it from the second. Which is also deduced from the same Definition as naturally as the former Case. *Exam.* Given 5 . 8, the third :*l* is 11; or given 8 . 5, the third :*l* is 2.

Universally to these *a*, *b*, a 3d is $b-a+b=2b-a$; or $b-\overline{a-b}=b-a+b=2b-a$.

SCHO-

SCHOLIUMS.

1. When a fourth :*l* is found, the four Terms will either be conjunct or disjunct, according as the Difference of the 2d and 3d is equal, or not, to that of the first and second; which is not considered in this *Problem*, nor in any of the following Propositions, where four Terms are mention'd.

2. A third or fourth :*l* is possible in all Cases, if the first Term is less than the second, but not if it is greater; for then the Difference may be greater than the second or third, and so cannot be subtracted; as in these 5.2, to which a third is impossible in real and positive Numbers; but if we bring in what are call'd, by the Algebraists, negative Numbers, or Numbers conceiv'd as less than nothing, then it is possible. For *Example:* To 5 and 2 I find a third, thus; Take the Difference $5-2=3$, and because it is greater than 2, and can't be subtracted from it, take $3-2=1$, and conceive this as it were subtracted from 0; then is $0-1$ the third Term sought, and the three stand thus, $5.2.0-1$. It is true indeed there is no such thing in Nature as a Number less than 0, and so this will be presently looked upon as chimerical. All I shall say here is, that for the common Applications it is of no Use; and for the Use that is made of it in Algebraical Reasonings and Calculations, I leave you to seek it in Books of that kind; yet thus much I shall shew you here, *viz.* that such an Expression answers in general to the Definition of Arithmetical Proportion, which requires only that the Difference of 5.2, and of $2,0-1$ be equal: and from the Nature of Subtraction this will be so, if that Difference (which is 3) added to $0-1$ make 2, which it does; that is, $0-1+3=2$. For if to 0 we add 3, the Sum is 3; from which taking one, the Remainder is 2, *i. e.* $0+3-1=2$; which is the same thing in effect as $0-1+3$; for the Signs $+$, $-$ signify only the adding the one, or subtracting the other from some first Quantity; and we may begin with either of these Operations, unless that first Quantity, as here 0, make the Subtraction really impossible; and then we begin with the Addition.

THEOREM I.

Example:
$a.b:c.d.$
$3.5:7.9.$
and
$3+9=5+7=12.$

If four Numbers are :*l*, thus, $a.b:c.d$, the Sum of the Extremes is equal to the Sum of the middle Terms, $a+d=b+c$.

DEMON. If the Difference were 0, the thing would be evident; and if it is not so, yet the greater Extreme *d* exceeds its adjacent Mean *c*, as much as the lesser Extreme *a* wants of its adjacent Mean *b*; whence the Truth propos'd is manifest.

Or *thus*; Let any Whole be represented by $a+d$; if you take any Quantity from *d* so as the Remainder be *c*, and add as much to *a*, so as the Sum be *b*, it is certain that, $b+c=a+d$, because we have added and subtracted equally from $\overline{a+d}$.

Or from *Prob.* 1. thus: To these three *a*, *b*, *c*, a fourth :*l* is $a-b+c$, or $b-a+c$; which added to *a*, the Sum is plainly $b+c$.

Or, *lastly*, Any four Numbers :*l* may be represented thus; $a.\overline{a+d}:b.b+d$. In which the *Theorem* is manifest, *viz.* $a+b+d=a+d+b$, being the same three Numbers on both Sides.

The *Reverse* of this *Theorem* is also true, *viz.* that of four Numbers taken in a certain Order, if the Sums of the Extremes and Means are equal, these Numbers are :*l*; that is, $a.b:c.d$ are :*l*, supposing $a+d=b+c$.

DEMONSTRAT. By equal Subtraction of *a* from both Sides, it is $d=b+c-a$. *Again*; By equal Subtraction of *c*, it is $d-c=b-a$; wherefore $a.b:c.d$, their Differences being equal.

Again;

Again; The contrary of this *Theorem* and its *Reverse* is also true, *viz.* if four Numbers are not :*l*, the Sums of the Extremes and Means are not equal: for if this were, the Numbers would be : *l* (by the Reverse); and if the Sums are not equal, the Numbers are not :*l*; for if they were, these Sums would be equal; by the *Theorem*.

Scholium. To save needless Repetition, I would have it observ'd once for all, That where-ever any *Theorem* and its *Reverse* are true, the Contraries are also true; for they plainly follow from the other. Of which we may make this *general Demonstration*: Let A and B represent two Propositions; then suppose that if A is true, B is also true; and reversly, if B is true, A is also true: Their Contraries are also true, *i. e.* if A is not true, neither is B; for if this were, the other would be so also, (by the Reverse of the first); and if B is not true, neither is A; for if this were, so would be the other, by the first. Now in all that follows, both in Arithmetical and Geometrical Proportions, where the Reverse of a *Theorem* is true, I shall mention it, and, if need be, demonstrate it; if it is not true, I shall shew it by one *Example*, which will be sufficient to make this appear; and therefore I shall never trouble you with mentioning the Contraries.

COROLLARIES.

I. If three Numbers, $a.b.c$ are :*l*, the Sum of the Extremes is equal to twice the Mean, or this is the half of that; *i. e.* $a+c=2b$: For by repeating the Mean thus, $a.b:b.c$, it is the same with the Case of four Numbers; or this Case may be also represented thus, $a.\ a+d.\ a+2d$; wherein the Truth propos'd is manifest; for the Sum of the Extremes is $2a+2d=2\times\overline{a+d}$. Or, by *Cor.* to *Prob.* 1. where we see that a 3d :*l* to a, b is $2b-a$, consequently $2b-a+a=2b$.

$a.b\ c$
$2.5.8$
and
$2+8=2\times5=10.$

The *Reverse* of this is also true; for if $a+c=2b$, then $b-a=c-b$; whence $a.b.c$ are :*l*.

II. Hence we have got another Rule for finding a third or fourth :*l*, *viz.*

1. For a third. Subtract the first from double the second, and the Remainder is the third. *Example*: To these 3.7. a third is 11. Thus; $2\times7=14$, then $14-3=11$.

2. For a fourth. Subtract the first from the Sum of the second and third, the Remainder is the fourth. *Example*: To these 12.5.8, a 4th is 1. Thus $5+8=13$, then $13-12=1$.

Problem II.

Of four Numbers :*l*, $a.b:c.d$, having the two Extremes a, d, and one Mean, to find the other.

Rule. From the Sum of the Extremes subtract the given Mean, the Remainder is the Mean sought. Thus, $\overline{a+d}-b=c$.

Demon. This follows from the last *Theorem*, and is evident in this Representation, $a.\overline{a+d}:b.\overline{b+d}$; wherein $a+b+d-b=a+d$.

Example:
$a.b:c.d$
$3.5:7.9$
Operation:
$3+9=12$, and
$12-5=7$, or
$12-7=5$.

Scholium. We may solve this Problem thus, *viz.* Take the Difference of the given Mean and its adjacent Extreme, and add or subtract it with the other Extreme, as the Case requires.

Prob-

PROBLEM III.

To find an Arithmetical Mean, b, *betwixt two given Numbers*, a *and* c.

RULE. Take half the Sum of the given Numbers, it is the Mean sought. Thus, $b=\frac{a+c}{2}$.

Example: $a=4.\ c=14.$ $b=9=\frac{4+14}{2}=\frac{18}{2}.$

DEMONSTR. This follows from the *Coroll.* last *Theorem*, or may be made evident thus; $a.\ a+d.\ a+2d$ are three Numbers $\div\, l$, and the Sum of the Extremes is $2a+2d$, whose half is $a+d$ the middle Term.

SCHOLIUM. We may solve this Problem thus; Take half the Difference of the given Numbers, add it to the lesser, or subtract it from the greater, and you have the Mean sought. The Reason is obvious.

THEOREM II.

IF four Numbers are $:l$, $a.b:c.d$, they will be so alternately, thus, $a.c:b.d$.

Example:
$2.6:3.7$
$2.3:6.7$

DEMONSTR. This is clear from *Theor.* 1, because the middle Terms being still the same, only in a different Order, their Sum is the same; and being equal to the Sum of the Extremes, hence they are $:l$, also in the alternate Order.

Or we may demonstrate the *Theorem* thus; Since $a.b:c.d$, then $a+d=b+c$, *Theor.* 1. take away a and also b from both Sides, and we have $d-b=c-a$; whence $a.c:b.d$.

LEMMA.

Part I. If two Numbers are added to, or subtracted from other two orderly, *i. e.* the lesser of the one with the lesser of the other, and the greater with the greater: The Sums will differ by the Sum, and the Differences by the Difference of the Differences of the given Numbers.

Example 1. 6.9 added to 5.7 the Sums are $6+5=11$, and $9+7=16$; then $16-11=5=9-6\ (=3)+7-5\ (=2)$.

Example 2. 5.7 subtracted from 8.13, the Remainders are $8-5=3$, and $13-7=6$; then $6-3=3=13-8\ (=5)-7-5\ (=2)$.

DEMONSTR. Let $a\ a+d$ and $b.\ b+c$ represent any given Numbers; their Sums, taken as proposed, are $a+b$, and $a+d+b+c$, whose difference is plainly $d+c$ the Sum of the given Differences. Again; Their Differences are $a-b$ and $a+d-b-c$, which we may also express $\overline{a-b}+\overline{d-c}$, or $\overline{a-b}-\overline{c-d}$, according as d or c is greatest; in which Expressions the Difference from $a-b$ is $d-c$, or $c-d$, the Difference of the given Differences d and c: For if d is greater than c, we must take it $\overline{a-b}+\overline{d-c}$, exceeding $\overline{a-b}$ by $d-c$: And if c is greater than d, we take $\overline{a-b}-\overline{c-d}$, wanting of $\overline{a-b}$, the Difference $c-d$.

Part II. If two Numbers are added to, or subtracted from other two in a contrary Order, (*viz.* the lesser with the greater) the Sums differ by the Differences, and the Differences by the Sums of the first Differences.

Ex. 1.	Diff.	*Ex.* 2.	Diff.	Universally:	Diff.
4 . 7	3	8 . 15	7	$a \,.\, a+d$	d
6 . 2	4	7 . 3	4	$b+c \,.\, b$	c
10 . 9	1	1 . 12	11	Sums $a+b+c \,.\, a+b+d$	$c-d$. or $d-c$.
				Diffs. $a-b-c \,.\, a+d-b$	$d+c$
				Or $b+c-a \,.\, b-a-d$	$c+d$.

Demonstr. For the Sums, the thing is plain in the Universal *Example* annex'd. For the Differences it may be a little further explained, thus; $a+d-b$ is plainly greater than $a-b-c$, and their Difference is $\overline{a+d-b}-\overline{a-b-c}=a+d-b-a+b+c=d+c$. Again; $b+c-a$ is greater than $b-a-d$, and their Difference is $\overline{b+c-a}-\overline{b-a-d}=b+c-a-b+a+d=c+d$.

Coroll. *to the 1st Part.* If two Numbers are equally multiplied or divided, the Products or Quotes differ by the like Multiple or aliquot Part of the given Difference: Because Multiplying and Dividing is but a repeated Adding and Subtracting. Or it may also appear thus; $an-bn=\overline{a-b}\times n$, and $\frac{a}{n}-\frac{b}{n}=\frac{a-b}{n}$.

Theorem III.

If four Numbers :*l* are added to, or subtracted from other four also :*l*, in order, the lesser with the lesser, and greater with the greater Term in the respective Couplets, the Sums or Differences are also :*l*, with a Difference which is the Sum or Difference of the given Differences (*Exam.* 1. and 2.) But if they are added or subtracted in a contrary Order, the Sums and Differences are also :*l*; but the Sums have a Difference equal to the Difference, and the Differences have a Difference equal to the Sum of the given Differences (*Exam.* 3. and 4.).

Example 1.	Diff.	*Example* 2.	Diff.	*Example* 3.	Diff.	*Example* 4.	Diff.
2 . 4 : 3 . 5	2	6 . 4 : 9 . 7	2	3 . 5 : 8 . 10	2	14 . 13 : 12 . 11	1
5 . 6 : 9 . 10	1	5 . 2 : 4 . 1	3	12 . 9 : 4 . 1	3	7 . 9 : 3 . 5	2
7 . 10 : 12 . 15	3	1 . 2 : 5 . 6	1	15 . 14 : 12 . 11	1	7 . 4 : 9 . 6	3

Demonstr. The universal *Reason* of both Parts of this *Theorem* is plainly contained in the preceeding *Lemma*: For the four Numbers being :*l*, the Sums or Differences are in the same Difference, which by the *Lemma* is according to what's here proposed.

COROLLARIES.

1. If four Numbers :*l*, or any Series $\div l$ are equi-multiplied or divided, the Products or Quotes are also :*l* or $\div l$, with a Difference which is the like Product or Quote of the given Difference: Thus, If $a.b:c.d$, then $an.bn:cn.dn$, and $\frac{a}{n}.\frac{b}{n}:\frac{c}{n}.\frac{d}{n}$.

2. If two Series $\div l$ are orderly added to or subtracted from one another, the Sums or Differences are $\div l$, with a Difference that is the Sum or Difference of the given Differences: And if they are added or subtracted in a contrary Order, they are also $\div l$; but the Sums differ by the Difference, and the Differences by the Sum of the given Differences.

Example

Exam. 1.	Diff.	*Exam.* 2.	Diff.	*Exam.* 3.	Diff.	*Exam.* 4.	Diff.
3. 5. 7. 9.11	2	3.6.9.12.15	3	1. 3. 5. 7. 9	2	11.13.15.17.19	2
2. 6.10.14.18	4	2.4.6. 8.10	2	14.11. 8. 5. 2	3	8. 7. 6. 5. 4	1
5.11.17.23.29	6	1.2.3. 4. 5	1	15.14.13.12.11	1	3. 6. 9.12.15	3

§. II. *Of Arithmetical* Progression.

Observe, *By the* Distance *of one Term of a Series from another, is meant, The Number of Terms from the one exclusive to the other inclusive; or, including both, it's the Number of Terms less* 1.

PROBLEM IV. *To raise an* Arithmetical Series *from a given Number, with a given Difference.*

RULE. Add the given Difference to, or subtract it from, the given Number; the Sum or Remainder is the 2d Term: To or from which add or subtract the same Difference, and you have the 3d Term; and thus you may go on to any Number of Terms, increasing; but decreasing, you can go no farther than till the last Term found is equal to or less than the Difference.

DEMONSTR. The *Reason* of this *Rule* is plainly contained in the Definition of Arithmetical Progression.

Example 1. Given, the 1st Term 3, and common Difference 2; the Series is 3.5.7.9.11.13. *&c.*

Example 2. The 1st Term 20, and Difference 4; a decreasing Series is 20.16.12.8.4.

SCHOLIUMS.

1. An Arithmetical Progression can be raised, increasing *in infinitum,* from any given Number, but not decreasing; for by continual lessening of any Number you must at last exhaust the whole.

2. If the two first Terms of a Series be called $a.b,$ the whole may be represented thus, $a.b.2b-a.3b-2a.4b-3a.$ *&c.* for these are the Expressions that arise by finding a 3d $:l$ to $a.b,$ and a fourth to $a.b,$ and the Term last found, and so on (by *Prob.* 1.) Where *observe,* That the Number multiplying b is the Distance of that Term from $a.$ But again;

3. If we call the lesser Extreme of a Series $a,$ the greater $l,$ and the common Difference d; then, by the Method of the present *Problem,* it's manifest that the Increasing Series from a is represented thus, $a.a+d.a+2d.a+3d.$ &c. to $.l.$ and the Decreasing Series thus, $l.l-d.l-2d.$ &c. to a: from which Representations these *Corollaries* are evident.

COROLLARIES.

1. If the lesser Extreme of a Series is the common Difference, the other Terms are the several Multiples of that Difference. Thus: If $a=d,$ then $a, a+d, a+2d,$ *&c.* is the same as this, $d, 2d, 3d,$ *&c.* Wherefore the Series of the Multiples of any Number is an Arithmetical Progression.

2. If o is the first Term, the second is the common Difference; and all the following are the Series of the Multiples of that Difference: Thus; o, $d, 2d, 3d,$ *&c.*

3. Betwixt o and any Multiple of a Number d, as nd, there can be put as many Arithmetical Means as the Multiplier leſs 1, or $n-1$; the common Difference being that Number d; as is manifeſt in this Series o, d, $2d$, $3d$.

4. Every Arithmetical Progreſſion, whoſe firſt Term is not o, has its ſeveral Terms equal to the Sums of the ſeveral Terms of ſuch a Series, added to the firſt Term of the given Series. Thus; In a, $a+d$, $a+2d$, *&c.* they are equal to the Sums of a added to each Term of this Series o, d, $2d$, *&c.*

5. As any Progreſſion may be thus repreſented, a, $a+d$, $a+2d$, *&c.* a being the leſſer Extreme, and d the common Difference, it is manifeſt that the Multiplier of d in every Term expreſſes the Diſtance of that Term from the firſt. And hence it follows immediately, that

6. Every Term is equal to the Sum of the leſſer Extreme, and ſuch a Multiple of the Difference d, whoſe Multiplier is the Diſtance of that Term from a. And *reverſly*, the leſſer Extreme is equal to the Difference of any greater Term, and ſuch a Multiple of the common Difference, whoſe Multiplier is the Diſtance of that Term from the leſſer Extreme. *Thus:* Let the Diſtance of any Term from a be m, that Term is $a+md$; which if we call l, then is $l=a+md$; and reverſly, $a=l-md$.

Scholium. Becauſe the greater Extreme of a Series is, by what's now ſhown, equal to $a+md$, or $a+\overline{n-1}\times d$, (m being the Diſtance of the Extremes $=n-1$, the Number of Terms leſs 1) and every Term below having the common Difference once leſs contained in it, or multiplied by a Number leſs by 1 than its Multiplier in the preceding greater Term; therefore a decreaſing Series which, when the greater Extreme is called l, we have ſeen repreſented thus, *viz.* l, $l-d$, $l-2d$, *&c.* may alſo be repreſented thus, $a+md$, $a+\overline{m-1}\,d$, $a+\overline{m-2}\,d$, *&c.* Or thus; $a+\overline{n-1}\,d$, $a+\overline{n-2}\,d$, $a+\overline{n-3}\,d$, *&c.* (becauſe $m=n-1$) going on ſo till the Multiplier of d be equal to 1; and then we have $a+d$, the Term next a.

7. But again more univerſally: From the ſame Expreſſion of a Series, it's manifeſt that the Difference of the Multipliers of d in any two Terms expreſſes the Diſtance of theſe two Terms; for in every Term aſcending, the Difference is taken once more than in the preceding; and therefore from any Term to any other, it's as many times oftner taken in the greater than in the leſſer, as their Diſtance expreſſes, *i. e.* the Difference of the Multipliers of d is their Diſtance. And hence it follows immediately, that

8. Any Term of an Arithmetical Progreſſion is equal to the Sum or Difference of any other Term; and ſuch a Multiple of the common Difference, whoſe Multiplier is the Diſtance of theſe Terms. Alſo the Difference of any two Terms is equal to ſuch a Multiple of the common Difference, whoſe Multiplier is the Diſtance of theſe Terms. For any Term being expreſſed $a+md$, (m the Diſtance of this Term from a) any greater Term muſt have a Multiple of d, whoſe Multiplier exceeds m by the Diſtance of theſe Terms (by the laſt); ſo that Diſtance being r, the greater Term is $a+\overline{m+r}\,d$. But $\overline{m+r}\,d=md+rd$. Hence $a+\overline{m+r}\,d=a+md+rd$. And reverſly, $a+\overline{m+r}\,d-rd=a+md$; and $a+\overline{m+r}\,d-a-md=rd$. Or this Truth may alſo be deduced from *Cor.* 6. Thus; Any Part of a Series, *i. e.* from any Term to any other, is ſtill an Arithmetical Progreſſion, whereof theſe two Terms are the Extremes; which being called a, L, and their Diſtance m, it's ſhewn that $a=L-md$, and $L=a+md$; and hence, *laſtly*, $L-a=md$.

Exam. In this Series, 3. 5. 7. 9. 11. 13. 15. If we compare 5 and 13, whoſe Diſtance is 4; then is $5=13-8$ $(=2+4)$ and $13=5+8$; alſo $13-5=8$.

Scholium. The immediate Uſe and Application of this laſt Truth we have in the Solution of theſe Problems.

1. Having

(1.) Having one Term of a Series, and the *common Difference*, to find a Term at any Diſtance from the given one, without finding all the intermediate ones; the Solution of which is plainly contained in this *Corol.* and is this; Multiply the common Difference by the given Diſtance, the Sum or Difference of this Product, and the given Term, is the Term ſought; ſo in the preceding *Exam.* $13 = 5 + 8$ $(= 2 \times 4)$ and $5 = 13 - 8$.

(2.) Having any two Terms and their Diſtance, to find any other Term at any Diſtance from either of the given Terms; which is ſolved thus: Take the Difference of the given Terms, and divide it by the given Diſtance, the Quote is the *common Difference*: By which we can find a Term at any Diſtançe from either of the given Terms by the Method of the preceding.

(3.) Of theſe three things, *viz.* the *common Difference*, the Diſtance of any two Terms, and the Difference of theſe two Terms, having any two, the third Term may be found; for any Term being called a, d the common Difference, and m the Diſtance of any greater Term from a, that Term is $= a + md$; and the Difference of a, and $a + md$, is md; which if we call M, then (1.) If d, m are given, M $(= md)$ is alſo known. (2.) If M and m are given, d is alſo known; for it is $= M \div m$. (3.) If M and d are given, m is known; for it is $= M \div d$.

9. The Sum of the Extremes (or of any other two Terms) of an Arithmetical Progreſſion, is equal to twice the leſſer added to the Product of their Diſtance and *common Difference*. Thus; The leſſer Extreme a, the greater l, the Diſtance m, and the *common Difference* d; I ſay, $a + l = 2a + md$; for $l = a + md$, by *Coroll. 6.* wherefore $a + l = a = a + md = 2a + md$.

SCHOLIUM. We ſhall apply this to ſome particular Progreſſions, as,

(1.) Suppoſe the leſſer Extreme 1, and the *common Difference* 2, as in this Series $1 : 3 : 5 : 7$, &c. (which we call the natural Series of odd Numbers) the Sum of the Extremes is always double the Number of Terms; which if we call n, then $a + l = 2n$; for the leſſer Extreme being 1, its double is 2, equal to the *common Difference*; therefore $a + l = 2a + md = 2 + 2m = 2 \times \overline{1 + m} = 2n$, becauſe $n = 1 + m$.

(2.) Suppoſe the leſſer Extreme is the *common Difference*, the Sum of the Extremes is equal to the Product of the *common Difference*, and Number of Terms $+ 1$. Thus; Since $a = d$, then $a + l = 2a + md = 2d + md = d \times \overline{2 + m} = d \times \overline{n + 1}$, becauſe $m = n - 1$; and therefore $m + 2 = n - 1 + 2 = n + 1$. Or ſee it thus; Such a Series is a, $2a$, $3a$, &c. na, and $a + na = a \times \overline{n + 1}$.

Yet more particularly, if the leſſer Extreme and Difference is 2, as in this Series $2 : 4 : 6 : 8 : 10$, &c. (which we call the natural Series of even Numbers); then is $a + l = 2 \times \overline{n + 1} = 2n + 2$; that is, double the Number of Terms $+ 1$; or the Sum of 2, and twice the Number of Terms.

10. The Difference of the Extremes, (or of any two Terms) is equal to the Product of their Diſtance by the common Difference; for $l = a + md$, therefore $l - a = md$.

THEOREM V.

IF there are two Arithmetical Series having the ſame common Difference, any two Terms in the one are $: l$, or have the ſame Difference with any two in the other, taken at the ſame Diſtance.

Exam. In this Series, 2. 4. 6. 8. 10. 12; and this, 5. 7. 9. 11. 13. 15. Theſe are $: l$, *viz.* $2.8 : 5.11$; and theſe $4.12 : 7 : 15$.

DEMON.

Demon. In each Pair the greater contains the lesser, and the *common Difference* taken the same Number of Times, (*i. e.* the same Multiple of *d*) because of the equal Distances.

Theorem VI.

Any two Terms of an Arithmetical Series are : *l*, with any other 2 taken at the same Distance.

Exam. In this Series, 3.5.7.9.11.13.15; these are : *l*, 3.7 : 11.15; and 3.9 : 5.11.

Again; If you take any 3 or more Terms equally distant from one another, they make a Progression, or continued Series; so in the preceding Series, 3.9.15, or 3.7.11.15.

Demon. The Reason is the same as in the last *Theorem*.

COROLLARIES.

1. If from the Sum of any two Terms you take any other Term, the Remainder is equal to a Term of the Series as far distant from one of the Terms added on the one hand, as the Subtractor is from the other of them on the other hand: For since any four Terms are : *l*, whereof the two lesser are at the same Distance as the two greater, (and consequently the least and that next to the greatest at the same Distance as the greatest and that next to the least,) therefore, from the Sum of the two middle Terms of these four, any of the Extremes being subtracted, the Remainder is the other Extreme. Or from the Sum of the Extremes, one of the Means being taken, leaves the other Mean; whence the *Coroll.* is manifest.

Observe again: If the Subtractor is one of the Extremes, *i. e.* lies on the same hand of (or is lesser or greater than) either of the Terms added, the Remainder will be the other Extreme, *i. e.* will lie on the opposite hand of (or be contrarily greater or lesser than) either of the Terms added; and consequently the Distance of the Remainder from the Subtractor will be the Sum of the Distances of both the Terms added from the Subtractor, (or from the Remainder it self, which is at the same Distance): But if the Subtractor is one of the Means, *i. e.* lies betwixt (or is less than one, and greater than the other of) the Terms added, the Remainder will be the other Mean, or lies also betwixt the Terms added; and consequently its Distance from the Subtractor is the Difference of the Distances of the Terms added from the Subtractor, (or from the Remainder, which is the same Distance.

Exam. In this Series, A.B.C.D.E.F.G.H, it's true that $C+F-A=H$. Because A.C : F.H, which is as far from F on the one hand, as A is from C on the other hand; and as far from A, as the Sum of the Distances of C and F from A. Again; $A+H-C=F$, which is as far from H on the one hand, as C is from A on the other; and as far from C, as the Difference of the Distances of A and H from C.

2. If any Term of a Series is doubled, and from that double another Term subtracted, the Remainder is a Term of the same Series, as far distant from the Term doubled on the one hand, as the Subtractor is on the other; and consequently the Remainder is twice as far from the Subtractor, as the Term doubled is. Hence reversly, the half of the Sum of any two Terms is equal to a Term in the middle of the Terms added, if there is such a middle Term. But however, this is true, That the Term which is in the middle betwixt any two Terms is the half of their Sum.

So in the preceding Series, $2D-B=F$, and $D=\frac{B+F}{2}$, because B.D.F are ÷ *l*.

3. Again; More universally, If any Term of a Series ÷ *l* is multiplied by any Number, and from the Product be subtracted the Product of another Term by a Multiplier less than the former by 1; the Remainder is a Term of the same Series, whose Distance from

that Term whose Multiple is subtracted is equal to the Product of these two, *viz.* the Number which multiplied the Term whose Multiple is the Subtrahend, and the Distance of the two given Terms from one another. The *Reason* will be plain from this *Example*: Let any two Terms of a Series be A . B; a Series continued from these is A . B . 2 B — A . 3 B — 2 A . 4 B — 3 A. *&c.* (*Schol.* Prob. 1. *&c.*) But by the two preceding *Coroll.* each of these Terms is a Term of any Series to which A, B, can belong; since 2 B — A is a 3d :*l* to A . B, and each of the rest a 4th to A . B, and the preceding. And here it's evident that each Term is twice, thrice, *&c.* as far from A as B is, according to the Multiplier of B.

4. If any three or more Terms are added together, and from the Sum be taken the Product of a Term lesser or greater than any of them, multiplied by a Number 1 less than the number of Terms added, the Remainder is a Term of the Series whose Distance from the Term whose Multiple is subtracted, is equal to the Sum of the Distances of the Terms added, from the same.

The *Reason* will be plain from the *Theorem*, thus: Let any Term of a Series be A, and any other two, both greater or both lesser than A, be B, C, then B + C — A is a Term of the Series as far from A, as the Sum of the Distances of B and C from A (by *Coroll* 1.) Add another Term D, the Sum is B + C + D — A, from which subtract A, the Remain. is B + C + D — 2 A; which is a Number found in the manner proposed, and (by *Cor.* 1.) is a Term of the Series as far distant from A, as the Sum of the Distances of the two Terms added, *viz*, $\overline{B + C - A}$ and D, which is the Sum of the Distances of the three Terms B + C + D. It's manifest that by adding another Term continually, and subtracting A at every Step, the same thing will still hold true: For at every Step there will be one Term more added, and A once more subtracted; so that the Multiplier of A will be still 1 less than the number of Terms added.

Or we may see this Truth in another manner without the *Theorem*, thus: Any Term of a Series may be called A, and if the common Difference of the Series be d, then all the Terms above A are $A + d$. $A + 2d$. $A + 3d$. *&c.* Suppose any three or more of these Terms are added together, and let n represent the Number of Terms added; also let m represent the Sum of all the Numbers which multiply d (*i. e.* the Sum of the Distances of the several Terms added, from A), it's manifest that their Sum will be $nA + md$: from which subtract $\overline{n - 1} \times A = nA - A$, the Remainder is $nA + md - nA + A = A + md$; which by the nature of an Arithmetical Series is such a Term of a Series whose lesser Extreme is A, and the Difference d, as that its Distance from A is equal to m, the Sum of the Distances of the Terms added. If the Series is $A . A - d . A - 2d$, *&c.* the Demonstration will proceed the same way.

SCHOLIUM. The immediate Use and Application of these *Corollaries* is in the Solution of the following Problems.

(1.) To find any Term of a Series having its Distance from the 1st Term, also the 1st Term, and any 2 other, the Sum or Difference of whose Distances from the 1st Term is equal to the Distance of the Term sought: The Solution of which is plainly contained in *Coroll.* 1. and need not be repeated.

2. To find any Term of a Series having the 1st, and any 3 or more others, the Sum of whose Distances from the 1st is equal to the Distance of the Term sought; as in *Coroll.* 4.

3. To find any Term of a Series, having the 1st, and any other whose Distance from it is an aliquot Part of the Distance of the Term sought; as in *Coroll.* 1. or 2.

4. To find any Term of a Series, having the first, and another whose Distance from the 1st is double the Distance of the Term sought.

Observe, If the Term sought is betwixt the Terms given, but not in the very middle, you have a Rule for solving this in *Prob.* 1. *Cor.* 8. *Sch.* 2.

THEOR.

Theorem VII.

In any Arithmetical Series, the Sum of the Extremes is equal to the Sum of any two mean Terms equally distant from the respective Extremes, (*i. e.* the lesser Mean from the lesser Extreme, and the greater from the greater; or contrarily) and to the double of the middle Term, where the Number of Terms is odd; *that is*, these Sums are all equal, *viz.* that of the Extremes, and of every two mean Terms equally distant from the Extremes; and the double of the middle Term, when the Number of Terms is odd.

Exam. In this Series, 3. 6. 9. 12. 15. 18. 21. 24. 27. These are equal, $3+27=6+24=9+21=12+18=2\times 15=30$.

Demon. This follows easily from the preceding, compared with *Theor.* 1. For the Terms, whose Sums are here said to be equal, are $:l$ by the preceding; and the Sums of the Extremes and Means, or double the middle Term, are equal by *Theor.* 1. *Cor.* 1. Thus, A. B. C. D. E. F. G being a Series $:l$, these are $:l$, A. B : F. G; hence $A+G=B+F$. Again; A. C : E. F, and $A+F=C+E$; and C. D. E being $:l$, $C+E=2D$.

Or we may shew this Truth by another Representation, as in the Margin; wherein A is the lesser Extreme, m the Distance of the Extremes, and d the *common Difference*; so that $A+md$ is the greater Extreme, (*Cor.* 6. *Prob.* 4.) And the same Series being continued from A, and $A+md$. Which we suppose carried equally on, *i. e.* to half of the Number of Terms, if that is even; and to the middle Term inclusive in both, if the Number of Terms is odd. You see, that as in the increasing Series d is once more added in every Step; so in the decreasing one, it is once more subtracted; and consequently the Sums of the correspondent Terms in the two Series must still be equal to the Sum of the Extremes, *viz.* $2A+md$. For any Whole being composed of two Parts, if we take away from the one, and add as much to the other, the whole, or Sum of these Parts continues still the same; so by constantly adding d to A, and subtracting it from $A+md$, the Sum remains equal.

$$\left.\begin{array}{ll} \textit{Lesser Extr.} & \textit{Greater.} \\ A & +A+md \\ A+d & +A+\overline{m-1}\,d \\ A+2d & +A+\overline{m-2}\,d \\ A+3d & +A+\overline{m-3}\,d \\ \&c. & \&c. \end{array}\right\} = 2A+md.$$

Or it may be more simply represented by making l the greater Extreme, and subtracting d continually from it; thus, $A.\ A+d.\ A+2d$, &c. $l-2d.\ l-d.\ l$, carrying each Part from A and l equally on, as before; where the same Truth is manifest from the same Principle of equal Addition and Subtraction.

SCHOLIUMS.

1. When a Series has an even Number of Terms, there are two Terms which we call the *Two middle Terms*; and then the *Theorem* may be expressed thus; The Sum of the two middle Terms, and of every two equally distant are equal: And we may see the same Truth also in this Representation, &c. $m-2d.\ m-d.\ m.\ n.\ n+d.\ n+2d.$ &c. Increasing and decreasing from the two middle Terms m, n.

2. Where the Series has an odd Number of Terms (*i. e.* a middle Term equally distant from both Extremes), then we may express the *Theorem* thus; The Double of the middle Term, and the Sums of every two Terms equally distant from it, are equal; and it may be represented thus; &c. $m-2d.\ m-d.\ m.\ m+d.\ m+2d.$ &c.

3. *Observe* also, That the Sum of any two Terms in a Series is equal to the Sum of any other two equally distant from the former two respectively; because the four are $:l$. Also Double of any Term is equal to the Sum of any two equally distant from it; or, Any Term is equal to the Half Sum of any equally distant from it.

4. Again

4. *Again*: When a Series has an even Number of Terms, tho' the two middle Terms are not in the continued Ratio of all the rest above and below, yet the Sum of the Extremes, and of every two Terms equally distant from them, will still be equal; for the four are $:l$ at least disjunctly, because of the *common Difference* and *equal Distance*.

General SCHOLIUM.

In every Arithmetical Progression these five things are considerable, *viz.* The two Extremes, the Common Difference, the Number of Terms, and the Sum: From which a Variety of Problems arise; whereof those are the chief and most useful, in which are given any three of these things, to find the other two; and these I shall next explain.

The Use of the Symbols *employed in the following* Problems.

$a =$ The lesser Extreme. $s =$ Sum of the Series. $n =$ Number of Terms.
$l =$ The greater Extreme. $d =$ Common Difference. $m = n - 1 =$ The Distance of the Extremes.

PROBLEM V.

Given the Extremes a, l, and Number of Terms n, To find the Difference d, and Sum s.

RULE 1. For d; The Difference of the Extremes divided by the Number of Terms less 1 Quotes of the common Difference sought, thus, $d = \frac{l - a}{n - 1}$.

Example: $a = 3$. $l = 15$ and $n = 7$. then is $d = 2 = \frac{15 - 3}{7 - 1} = \frac{12}{6}$, as in this Series 3 . 5 . 7 . 9 . 11 . 13 . 15.

DEMONSTR. In *Cor.* 10. *Probl.* IV. it's shewn that $d \times \overline{n - 1} = l - a$. and dividing equally by $n - 1$, it is $d = \frac{l - a}{n - 1}$.

2. For s; multiply the Sum of the Extremes by the Number of Terms, and take half of the Product, it's the Sum: Thus $s = \frac{\overline{a + l} \times n}{2}$.

Example: $a = 3$. $l = 15$. $n = 7$. then is $s = 63 = \frac{\overline{3 + 15} (= 18) \times 7}{2} = \frac{126}{2}$, as in the preceding Series.

DEMONSTR. 1. If the Number of Terms is *even*, (*i.e.* a Multiple of 2.) then the Sums of the Extremes, and of every Pair of Means equally distant from the Extremes, are equal (*Theor.* 7.) But all these equal Sums together make the total Sum; and it's evident there are as many of these equal Sums as the half Number of Terms; since each Sum takes in two Terms; therefore the Sum of the Extremes (or any one of these equal Sums) being multiplied by the Number of Terms, produces double the total Sum, and consequently its half is the Sum sought.

2. If the Number of Terms is odd (or not a Multiple of 2), then there is a middle Term, and an equal Number on each hand, which Number is plainly the half of $n - 1$, (the middle Term being excluded) and the Sum of all, excluding the middle, is by the former Reasoning, $\frac{\overline{a + l} \times \overline{n - 1}}{2}$: But the middle Term is $= \frac{a + l}{2}$ (*Theor.* 7.) which

being added to the Sum of the other Terms, makes the Total, *viz.* $\frac{\overline{a+l}\times\overline{n-1}}{2}+\frac{a+l}{2}$ $=\frac{\overline{a+l}\times\overline{n-1}+\overline{a+l}}{2}=\frac{\overline{a+l}\times\overline{n-1+1}}{2}=\frac{\overline{a+l}\times n}{2}$.

Or the whole *Demonstration* may be made, without distinguishing whether the Number of Terms is even or odd, thus; Any Series increasing may be represented; $a.a+d.a+2d$, &c. and the same Series decreasing may be taken thus, $l.\,l-d.\,l-2d$, &c. And these two representing the same Series, only in a different Order, have equal Sums; and therefore the Sum of both together is double the Sum of either. Again; It's obvious, that adding their corresponding Terms, the Sums are constantly the same, *viz.* $a+l$. and the Number of them being n, therefore the total of these equal Sums is $\overline{a+l}\times n=2S$, whence $s=\frac{\overline{a+l}\times n}{2}$.

$$a.a+d.a+2d.\&c.=s$$
$$l.\,l-d.\,l-2d.\&c.=s$$
$$a+l.\,a+l.\,a+l.\&c.\ 2s=$$
$$\overline{a+l}\times n.\ \text{hence}\ s=\frac{\overline{a+l}\times n}{2}$$

Scholium. In a Series of an even Number of Terms we may take the Sum of the Extremes, and multiply by half the Number of Terms; or half the Sum of the Extremes by the Number of Terms; for all these ways make the same Sum; thus, $\frac{\overline{a+l}\times n}{2}=\overline{a+l}\times\frac{n}{2}=\frac{a+l}{2}\times n$.

COROLLARIES.

1. Betwixt any two different Numbers a, l, we can put any Number of Arithmetical Means; because l being greater than a, and n greater than 1, it follows, that $d=\frac{l-a}{n-1}$ is always possible, however great n be. But then also *observe*, That betwixt two Integers these Means will not in every Case be Integers. And to make them all so, the greatest Number of Means cannot be a greater Number than the Difference less 1 betwixt the Extremes, or $l-a-1$. The Reason is plain; for to the lesser Term a, there can be successively joined as many Units as are in the Difference betwixt l and a; and when the last of these Units is added, the Sum will be equal to l; consequently the preceding Sums are in Number equal to $l-a-1$; and because they differ all by 1, therefore they make with the Extremes a and l an Arithmetical Progression; and the greatest that can be in Integers, because the Difference is the smallest, *viz.* 1.

2. The Sum of the natural Series, 1.2.3.4, *&c.* is equal to half the Product of the last Term multiplied into the next greater Number, because that next is the Sum of the Extremes, and the greatest Extreme is the Number of Terms: So $1+2+3+4+5=5\times 6\div 2$.

3. In any Series whose Number of Terms is odd, the Sum is equal to the Product of the Number of Terms multiplied into the middle Term; because the middle Term is half the Sum of the Extremes.

4. From this *Prob.* and *Cor.* 9. *Prob.* 4. compared, these things follow, *viz.*

(1.) The Sum of that Series, whose lesser Extreme is 1, and the Difference 2; as 1.3.5.7.*&c.* (which is the natural Series of odd Numbers) is equal to the Square of the Number of Terms. For in this Case, $a+l=2n$ (*Cor.* 9. *Prob.* 4.) and hence $\frac{a+l}{2}=n$, and $s=\frac{\overline{a+l}\times n}{2}=nn$.

(2.) The

(2.) The Sum of any Series, whoſe leſſer Extreme and Difference are equal, is equal to half the Product of theſe Factors, *viz.* the common Difference, and the Sum of the Number of Terms and its Square; thus, $s = \frac{dn^2 + dn}{2}$. For in this Caſe, $a + l = dn + d$ (*Cor.* 9. *Prob.* 4.) therefore $\overline{a+l} \times n = dn^2 + dn$. Hence $s = \frac{\overline{a+l} \times n}{2} = \frac{dn^2 + dn}{2}$. Again; Particularly if $a = d = 2$; as in this Series, 2. 4. 6. 8, *&c.* (which makes the natural Series of even Numbers; for it comprehends all the Multiples of 2.) The Sum is equal to the Sum of the Number of Terms and its Square; thus, $s = n^2 + n$; for $dn^2 + dn = 2n^2 + 2n$; therefore $s = \frac{dn^2 + dn}{2} = \frac{2n^2 + 2n}{2} = n^2 + n$.

PROBLEM VI.

Having the Extremes a, l, and Difference d; to find the Number of Terms n, and Sum s.

RULE 1. For n. Divide the Difference of the Extremes by the common Difference, the Quote is the Number of Terms leſs 1. Thus, $n - 1 = \frac{l - a}{d}$; therefore $n = \frac{l-a}{d} + 1 = \frac{l - a + d}{d}$; *i. e.* to the greater Extreme add the *common Difference*, and from the Sum take the leſſer Extreme, and divide the Remainder by the *common Difference*, the Quote is the Number of Terms.

Exam. $a = 3$, $l = 15$, $d = 2$; then is $n = 2 = \frac{15 + 2 - 3}{2} = \frac{14}{2}$.

DEMON. By *Cor.* 10. *Prob.* 4. $d \times \overline{n - 1} = l - a$; and dividing each by d, it is $n - 1 = \frac{l-a}{d}$. Whence $n = \frac{l-a}{d} + 1 = \frac{l - a + d}{d}$.

2. For s. apply $a. l. n.$ By *Prob.* 5. thus, $s = \frac{\overline{a+l} \times n}{2}$. Or without finding n, apply the given Numbers, thus: Add the Difference of the Squares of the Extremes, to the Product of the Sum of the Extremes, by the common Difference; this Sum divided by double the *common Difference*, the Quote is the Sum ſought; thus, $s = \frac{\overline{a+l} \times d + l^2 - a^2}{2d}$.

Exam. $a = 3$, $l = 15$, $d = 2$; then is $s = 63 = \frac{\overline{3+15} \times 2 + 225 - 9}{2 \times 2} = \frac{36 + 216 = 252}{4}$.

DEMON. Inſtead of n take its Equal above found, $\frac{l - a + d}{d}$; and ſubſtitute this inſtead of n in the other Rule, *viz.* $s = \frac{\overline{a+l} \times n}{2}$, and it is $s = \frac{a+l}{2} \times \frac{l - a + d}{d} = \frac{\overline{a+l} \times d + l^2 - a^2}{2d}$; for $\overline{a+l} \times \overline{l - a + d} = al - a^2 + ad + l^2 - al + ld = al - al + ld + ad + l^2 - a^2 = ld + ad + l^2 - a^2 = \overline{a+l} \times d + l^2 - a^2$.

PROBLEM VII.

Having one of the Extremes a or l, with the Number of Terms n, and Difference d; to find the other Extreme l or a, and the Sum.

Rule 1. For l or a: Multiply the Difference by Number of Terms less 1; add the Product to the given Extreme if it's a, but subtract if it's l; the Sum or Remainder is the other Extreme. Thus, $l = d \times \overline{n-1} + a$, and $a = l - d \times \overline{n-1}$.

Exam. $a=3$, $n=7$, $d=2$; then is $l=15=2\times 6+3$.

Demon. This Rule is expresly contained in *Coroll.* 6. *Probl.* IV.

2. For s apply a, l, n, by *Prob.* 5. Or without finding n proceed thus: Multiply the given Extreme by double the Number of Terms; to the Product add (if the given Extreme is a), or subtract (if it's l) the Product of the common Difference multiplied by the Difference betwixt the Number of Terms and its Square: the half of this Sum or Difference is the Sum sought. Thus; $s = \frac{2na + d \times \overline{nn-n}}{2} = \frac{2nl - d \times \overline{nn-n}}{2}$, or $\frac{2nl - dnn + dn}{2}$.

So in the preceding, $s = 63 = \frac{2\times 7\times 3 + 2\times \overline{49-7}}{2}$, when $a=3$ is given.

Demon. Take a or l, as they are expressed in the first Part, *viz.* $l - d\times\overline{n-1}$ for a, and $a + d\times\overline{n-1}$ for l. Substitute these instead of a, l in this Rule, $s = \frac{\overline{a+l}\times n}{2}$. Thus, Since $a = l - d\times\overline{n-1}$, then $a + l = l - d\times\overline{n-1} + l = 2l - d\times\overline{n-1}$, and $\overline{a+l}\times n = 2ln - d\times\overline{n-1}\times n = 2ln - d\times\overline{nn-n}$. Consequently, $s = \frac{\overline{a+l}\times n}{2} = \frac{2ln - d\times\overline{nn-n}}{2}$; which is the Rule when l is given. Again; Since $l = a + d\times\overline{n-1}$, then $a + l = 2a + d\times\overline{n-1}$, and $\overline{a+l}\times n = 2an + d\times\overline{nn-n}$. Hence, $s = \frac{\overline{a+l}\times n}{2} = \frac{2an + d\times\overline{nn-n}}{2}$, the Rule when a is given.

Scholium. In the preceding Problems, we have found d by means of a, l, n. Also n by means of a, l, d, and a by means of d, n, l. And here it may be useful to observe a Mistake of a very considerable Author, Tacquet, who gives Rules for finding d, or n, or a, by means of only 2 of the given things in the preceding Problems. His Rules are these: d is the integral Part of this Quote $\frac{l}{n-1}$, and a is what remains over the Division. Again; $n-1$ is the integral Part of this Quote $\frac{l}{d}$; and a what remains over the Division. Which Rules he founds upon this, that $l = a + d\times\overline{n-1}$. Whence he concludes, that $\frac{l}{d} = n-1$, and a remaining over; also that $\frac{l}{n-1} = d$, and a remaining. But you'll easily perceive that these Conclusions are not true universally, and can be so only in a particular Case, *viz.* when a is less than d, or than $n-1$; yet he delivers them as general Rules without Limitation, at least without mentioning it, if he did observe any. The Reason and Necessity of this Limitation I thus prove: If $d\times\overline{n-1}$ is divided by either Factor d, or $\overline{n-1}$, the Quote will be the other of them; and if to the Product $d\times\overline{n-1}$ we had first added another Number a less than the Divisor, the Quote would be the same as before, and the Remainder would be the Number added. But if a the Number added, is equal to, or greater than the Divisor, it's plain that the Quote would be greater than the other

other Factor, becauſe it's contained in the Number added; and the Remainder would not be the Number added, but leſs, becauſe it muſt be leſs than the Diviſor, which here is leſs than the Number added. Take this particular *Example:* $5 . 7 : 9 : 11 : 13$; whoſe greater Extreme 13 being divided by 4 $(= n - 1)$ the Quote is 3 and 1 over; yet the common Difference of the Series is 2, and the leſſer Extreme 5.

What can be done by means of two things given, you'll learn afterwards.

PROBLEM VIII.

Having the Extremes, a, l, and Sum s; to find the Number of Terms n, and Difference d.

RULE 1. For n divide double the Sum by the Sum of the Extremes, the Quote is the Number of Terms, thus; $n = \frac{2s}{a + l}$.

Example: $a = 3 . l = 15 . s = 63$. then is $n = 7 = \frac{2 \times 63}{3 + 15} = \frac{126}{18}$.

DEMONSTR. By *Probl.* V. $s = \frac{\overline{a + l} \times n}{2}$; hence, multiplying both by 2, it is $2s = \overline{a + l} \times n$; then, dividing both by $a + l$, it is $n = \frac{2s}{a + l}$.

2. For d apply $a . l . n$. by *Probl.* V. Or without finding n, work thus; Take the Squares of the Extremes, and divide the Difference of theſe Squares by the Difference betwixt the Sum of the Extremes, and double the given Sum, the Quote is the common Difference; thus, $d = \frac{l^2 - a^2}{2s - a - l}$.

Example: $a = 3 . l = 15 . s = 63$; then is $d = 2 = \frac{15 \times 15 - 3 \times 3}{2 \times 63 - 3 - 15} = \frac{225 - 9}{126 - 18}$.

DEMON. Inſtead of n, take its Equal $\frac{2s}{a + l}$, and put in the Rule of *Prob.* V. *viz.* $d = \frac{l - a}{n - 1}$ thus, $\frac{2s}{a + l} - 1 = \frac{2s - a - l}{a + l}$, therefore $n - 1 = \frac{2s - a - l}{a + l}$, and $\frac{l - a}{n - 1} = \overline{l - a} \div \frac{2s - a - l}{a + l} = \frac{\overline{l - a} \times \overline{a + l}}{2s - a - l} = \frac{l^2 - a^2}{2s - a - l}$.

PROBLEM IX.

Having one Extreme a, or l, with the Sum s, and Number of Terms n, To find the other Extreme l or a, and the Difference d.

RULE 1. For l or a, divide double the Sum by the Number of Terms, and from the Quote ſubtract the given Extreme, the Remainder is that ſought. Thus, $l = \frac{2s}{n} - a$, and $a = \frac{2s}{n} - l$. Or thus: From double the Sum take the Product of the Number of Terms and given Extreme, and divide the Remainder by the Number of Terms, the Quote is the Extreme ſought; thus, $l = \frac{2s - an}{n}$, and $a = \frac{2s - ln}{n}$.

Example: $a = 3 . s = 63 . n = 7$. then is $l = \frac{2 \times 63 - 3 \times 7}{7} = 15$; and if $l = 15$ is given, then $a = 3 = \frac{2 \times 63 - 7 \times 15}{7}$.

DEMON.

DEMONSTR. By *Probl.* V. $s = \frac{\overline{a+l} \times n}{2}$: Hence $2s = \overline{a+l} \times n$, and $\frac{2s}{n} = a + l$. and lastly, $a = \frac{2s}{n} - l = \frac{2s - ln}{n}$, and $l = \frac{2s}{n} - a = \frac{2s - an}{n}$.

2. For d. apply a, l, n, by *Probl.* V. or without finding n, do thus; Take the Difference betwixt Double the Sum and Double the Product of the Number of Terms by the given Extreme; divide this by the Difference betwixt the Number of Terms and its Square; the Quote is the Difference sought; thus, $d = \frac{2s - 2an}{nn - n} = \frac{2ln - 2s}{nn - n}$.

Example: $a = 3 . n = 7 . s = 63$. then is $d = 2 = \frac{2 \times 63 - 2 \times 3 \times 7}{7 \times 7 - 7} = \frac{126 - 42}{49 - 7}$.

DEMONSTR. By the preceding Rule $a = \frac{2s - ln}{n}$: Substitute this instead of a in the *Rule* of *Probl.* V. *viz.* $d = \frac{l - a}{n - 1}$; thus, $l - a = l - \frac{2s - ln}{n} = \frac{ln - 2s + ln}{n} = \frac{2ln - 2s}{n}$. Therefore $\frac{l - a}{n - 1} = \frac{2ln - 2s}{n} \div \overline{n - 1} = \frac{2ln - 2s}{nn - n}$, the Rule for d when l is given.

Again; $l = \frac{2s - an}{n}$: hence $l - a = \frac{2s - an}{n} - a = \frac{2s - 2an}{n}$, therefore $d = \frac{l - a}{n - 1} = \frac{2s - 2an}{n} \div \overline{n - 1} = \frac{2s - 2an}{nn - n}$.

COROLL. We learn here how to find the Sum of the Extremes by means of the Sum of the Series, and the Number of Terms; thus, $a + l = \frac{2s}{n}$, as we see above; for $a = \frac{2s}{n} - l$, and $l = \frac{2s}{n} - a$. so that $a + l = 2 \times \frac{2s}{n} - a - l$. whence $a + l = \frac{2s}{n}$.

PROBLEM X.

Having the Sum of the Series s, the Difference d, and Number of Terms n, To find the Extremes a, l.

RULE. By the Sum and Number of Terms find the Sum of the Extremes, as in the last *Corollary*, viz. $a + l = \frac{2s}{n}$; then by d and n find the Difference of the Extremes, *viz.* $l - a = d \times \overline{n - 1}$ (*Cor.* 10. *Prob.* IV.) Lastly, having the Sum and Difference of the Extremes, find the Extremes thus: To the half of their Sum add half the Difference, the Sum is the Greater Extreme; And from the half Sum take the half Difference, the Remainder is the Lesser Extreme; thus, $l = \frac{a + l}{2} + \frac{l - a}{2} = \frac{s}{n} + \frac{d \times \overline{n - 1}}{2}$; for $a + l = \frac{2s}{n}$; therefore, $\frac{a + l}{2} = \frac{s}{n}$, and $l - a = d \times \overline{n - 1}$, and $\frac{l - a}{2} = \frac{d \times \overline{n - 1}}{2}$. Then $a = \frac{a + l}{2} - \frac{l - a}{2} = \frac{s}{n} - \frac{d \times \overline{n - 1}}{2}$: which Expressions being reduced to a more simple Form, they are equal to these, *viz.* $l = \frac{2s + dnn - dn}{2n}$, and $a = \frac{2s - dnn + dn}{2n}$; in which Terms also the *Rule* may be expressed.

Exam-

Example: $s = 63 . n = 7 . d = n$, then is $a = 3 = \frac{63}{7} - \frac{2 \times 6}{2} = 9 - 6$, and $l = 15 = \frac{63}{7} + \frac{2 \times 6}{2} = 9 + 6$.

DEMONSTR. The laſt Part of the *Rule* only wants to be demonſtrated; and becauſe it is a general Truth of frequent Uſe, I ſhall put it by itſelf in the Form of a

LEMMA.

The half Sum of two Numbers added to their half Difference makes the greater of the two, and their Difference makes the leſſer of the two.

Example: The Sum of two Numbers being 18, and their Difference 12, the Greater of them is $15 = 9 + 6$; and the Leſſer is $3 = 9 - 6$.

DEMONSTR. The two Numbers being a, l, the half Sum is $\frac{a+l}{2}$, and half Difference $\frac{l-a}{2}$, then is $\frac{a+l}{2} - \frac{l-a}{2} = \frac{a+l-l+a}{2} = \frac{2a}{2} = a$, and $\frac{a+l}{2} + \frac{l-a}{2} = \frac{a+l+l-a}{2} = \frac{2l}{2} = l$.

PROBLEM XI.

Having one Extreme a, or l, with the Sum s, and Difference d, To find the other Extreme l or a, and Number of Terms n.

RULE I. For the Extreme ſought: If it's the Leſſer, a, then take the Square of the Greater Extreme (l^2), to which add the Product of the Extreme by the common Difference (dl); and to this Sum again add the 4th Part of the Square of the Difference, $\left(\frac{dd}{4}\right)$ from which Sum ſubtract the Product of twice the Difference by the given Sum of the Series ($2ds$), out of which Remainder, *viz.* $l^2 + dl + \frac{dd}{4} - 2ds$, extract the Square Root; to which Root add half the common Difference, or $\frac{d}{2}$; the Sum is the Leſſer Extreme; thus, $a = \overline{l^2 + dl + \frac{dd}{4} - 2ds}|^{\frac{1}{2}} + \frac{d}{2}$.

For the Greater Extreme; To the Square of the Leſſer add the Product of the Sum into twice the Difference; and to this Sum again add the 4th Part of the Square of the Difference; from which Sum ſubtract the Product of the given Extreme and Difference; then take the Square Root of the Remainder, from which take half the Difference; the Remainder is the Greater Extreme; thus, $l = \overline{a^2 + 2ds + \frac{dd}{4} - ad}|^{\frac{1}{2}} - \frac{d}{2}$.

DEMONSTR. By *Prob.* 7. $l = a + dn - d$, and by *Prob.* 8. $n = \frac{2s}{a+l}$; ſubſtitute this for n in the former, and it is $l = a + \frac{2sd}{a+l} - d$, which being reduced is $\frac{aa + al + 2sd - ad - dl}{a+l} = l$. Multiply both by $a + l$, and it is $aa + al + 2ds - ad - dl = l^2 + al$. Subtract al from both, and it is $aa + 2ds - ad - dl = l^2$. Add dl to both, it is $aa + 2ds - ad = l^2 + dl$. wherefore (by *Prob.* 6. *Ch.* II. §. 2. *Book* III.) $l = \overline{aa + 2ds + \frac{dd}{4} - ad}|^{\frac{1}{2}} - \frac{d}{2}$. which is the Rule for l.

Again;

Again; By *Prob.* VII. $a=l-dn+d$, and by *Prob.* VIII. $n=\frac{2s}{a+l}$; therefore $a=l-\frac{2ds}{a+l}+d$; whereby we come to this, $a^2+al=al+l^2-2ds+ad+dl$: Take away al, and also ad from both, then $a^2-ad=l^2-2ds+dl$, and (by *Prob.* 6. *Ch* II. §. 2. *Book* III.) $a=\overline{l^2+dl+\frac{dd}{4}-2ds}|^{\frac{1}{2}}+\frac{d}{2}$, the Rule for a.

Example: $a=3.s=63.d=2.$ then is $l=15$. for $a^2=9.2ds=2\times2\times63=252.\frac{dd}{4}=1.ad=6.$ so that $a^2+2ds+\frac{dd}{4}-ad=9+252+1-6=262-6=256$. whose Square Root is 16. from which take $\frac{d}{2}=1$; The Remainder is $15=l$.

2. For the Number of Terms, n: If a is given, then from twice the lesser Extreme, $(2a)$ take the Difference (d); divide the Remainder $(2a-d)$ by the Difference: Then take the 4th Part of the Square of this Quote, to which add the Quote of twice the Sum $(2s)$ divided by the common Difference $\left(\frac{2s}{d}\right)$; Out of that Sum $\left(\frac{2s}{d}+\frac{1}{4}\right.$ of the Square of $\left.\frac{2a-d}{d}\right)$ extract the Square Root; from which take half of the first Quote (*viz.* $\frac{1}{2}$ of $\left.\frac{2a-d}{d}\right)$; the Remainder is the Number of Terms sought: Thus, for Brevity, call $\frac{2a-d}{d}=b$; then is $n=\overline{\frac{2s}{d}+\frac{b^2}{4}}|^{\frac{1}{2}}-\frac{b}{2}$.

If l is given, Take the Sum of twice l and d, which divide by d, and call this Quote $b\left(=\frac{2l+d}{d}\right)$; from the $\frac{1}{4}$ of the Square of this, $\left(viz.\ \frac{bb}{4}\right)$ take $\frac{2s}{d}$ and extract the Square Root of the Remainder *viz.* $\overline{\frac{bb}{4}-\frac{2s}{4}}|^{\frac{1}{2}}$. Take the Sum or Difference of this Root, and $\frac{b}{2}$, and one or the other is the Number of Terms: Which Rule is expressed thus, $n=\frac{b}{2}\pm\overline{\frac{b^2}{4}-\frac{2s}{d}}|^{\frac{1}{2}}$.

Demon. By *Prob.* 6. $n=\frac{l-a+d}{d}$, and (by *Prob.* 9.) $l=\frac{2s-an}{n}$. Hence $l-a+d=\frac{2s-an}{n}-a+d=\frac{2s-2an+nd}{n}$. Consequently $n=\left(\frac{l-a+d}{d}\right)=\frac{2s-2an+nd}{nd}$. Multiply both by nd, and it is $nnd=2s-2an+nd$; add $2an$, and subtract nd from both, and we have $nnd+2an-nd=2s$. Divide each Member by d, and it is $n^2+\frac{2a-d}{d}\times n=\frac{2s}{d}$. And calling $\frac{2a-d}{d}=b$, it is $n^2+bn=\frac{2s}{d}$. Whence (by *Prob.* 6. *Chap.* 2. §. 2. *B.* III.) we have $n=\overline{\frac{2s}{d}+\frac{b^2}{4}}|^{\frac{1}{2}}-\frac{b}{2}$, the Rule for n when a is given.

Again; By *Prob.* 9. $a=\frac{2s-ln}{n}$. Hence $l-a=l-\frac{2s-ln}{n}=\frac{ln-2s+ln}{n}=\frac{2ln-2s}{n}$, then $l-a+d=\frac{2ln-2s}{n}+d=\frac{2ln-2s+dn}{n}$, as above; therefore $n\ (=\frac{l-a+d}{d}$

as

as above) $= \frac{2ln - 2s + dn}{nd}$. Multiply both by nd, and it is $nnd = 2ln - 2s + dn$; add $2s$, and then ſubtract nnd from both Sides, and we have $2s = 2ln + dn - nnd$. Divide all by d, and it is $\frac{2s}{d} = \frac{2l+d}{d} \times n - n^2$. Call $\frac{2l+d}{d} = b$, then it is $\frac{2s}{d} = bn - n^2$; and (by *Prob. 6. Chap. 2. §. 2. Book* III.) $n = \frac{b}{2} \pm \overline{\frac{b^2}{4} - \frac{2s}{d}}|^{\frac{1}{2}}$, the Rule for n when l is given.

SCHOLIUM. Theſe Rules are tedious both in the Inveſtigation and Application: But there is another Method of ſolving the Problem, which tho' it is only by Trials, yet it proceeds directly and certainly to the Anſwer; and is rather eaſier than the former Work, and therefore I ſhall here explain it.

Another RULE.

By *Prob.* I. raiſe a Series from the given Extreme and Difference; and take the Sum of the Series gradually as it riſes, continuing this Operation till the Sum is equal to the given Sum; and the Series ſo raiſed will ſhew both the Number of Terms, and the Extreme ſought; which is the laſt Term found in the Series.

The Reaſon of this Rule will be obvious from one *Example.* Suppoſe $a = 4$, $d = 3$, $s = 91$. In the annext Operation, you ſee in the upper Line the given Extreme 4, and the Difference 3 continually added. In the ſecond Line are the Terms of the Progreſſion formed by that continual Addition; and in the third Line are the ſeveral Sums of the preceding Series taken continually from the Beginning, by adding the next Term to the preceding Sum. Whence we ſee in the preſent *Example*, that the Extreme ſought is 22, and the Number of Terms 7.

$$\begin{array}{ccccccc} 4+ & 3+ & 3+ & 3+ & 3+ & 3+ & 3 \\ 4. & 7. & 10. & 13. & 16. & 19. & 22 \\ 4. & 11. & 21. & 34. & 50. & 69. & 91 \end{array}$$

Obſerve, That the Tediouſneſs of this Method, when the Number of the Terms is great, may be relieved by the following Means, *viz.* Take any Number for n at a gueſs, (in which to prevent being too wide of the Truth, have a regard to the given Numbers); then by this Number n, with the given d and a or l, find the other Extreme, and the Sum; and if this Sum differs from that given, begin at the Extreme laſt found, and raiſe a Series, increaſing or decreaſing, as the Caſe requires, till you find a Sum equal to the given one. For *Example:* Suppoſe $a = 3$, $d = 4$, $s = 406$; I gueſs $n = 12$, and hereby find $l = 47$ $(= 4 \times 11 + 3)$ and $s = 300$ $(= \overline{47 + 3} \times 6)$; which being leſs than 406, I begin at 47, and adding the Difference 4 till the Sum is equal to 406, I find that this happens upon adding that Difference twice, *i. e.* that two Terms more with the 47 make the Sum given; whence 'tis certain that 14 is the Number of Terms, (for there were 12 to bring it to 47, and 2 now added) and 55 the greater Extreme.

$$\begin{array}{ccc} 47+ & 4+ & 4 \\ & 51. & 55 \\ 300. & 351. & 406 \end{array}$$

SCHOLIUMS.

I. For the more convenient and ready Uſe of the laſt ſeven *Problems*, we ſhall put them all in a Table, that they may appear in one View; expreſſed ſimply by their Characters, whoſe Signification I ſhall repeat:

a = leſſer Extreme.	n = Number of Terms.	s = Sum of the whole Series.
l = greater Extreme.	d = the common Difference.	

TABLE *of the preceding Seven* Problems.

Probl.	Given.	Sought.	Sotutions.
5	$a, l, n.$	$d, s.$	$d = \frac{l-a}{n-1}$. $s = \frac{\overline{a+l}\,n}{2}$.
6	$a, l, d.$	$n, s.$	$n = \frac{l-a+d}{d}$. s, as above; or thus, $s = \frac{ad+ld+l^2-a^2}{2d}$.
7	a or . n . d l	l or . s a	$l = a+d\overline{n-1}$. $a = l-d\overline{n-1}$. s, as above; or thus, $s = \frac{2an+dn^2-dn}{2}$ $s = \frac{2ln-dn^2+dn}{2}$
8	$a, l, s.$	$n. d.$	$n = \frac{2s}{a+l}$ d, as above; or thus, $d = \frac{l^2-a^2}{2s-a-l}$
9	a or . n . s l	l or . d a	$l = \frac{2s-an}{n}$ $a = \frac{2s-ln}{n}$ d, as in the 5th; or thus, $d = \frac{2s-2an}{n^2-n}$ $d = \frac{2ln-2s}{n^2-n}$
10	$d . n . s.$	$a. l.$	$a = \frac{2s-dn^2+dn}{2n}$. $l = \frac{2s+dn^2-dn}{2n}$
11	a or . d . s l	l or . n a	The Solution of this is by raiſing a Series from the given Extreme with the given Difference, till the Sum is equal to that given. Or thus, $a = \overline{l^2+dl+\frac{dd}{4}-2ds}\vert^{\frac{1}{2}} + \frac{d}{2}$ and $l = \overline{a^2+2ds+\frac{dd}{4}-ad}\vert^{\frac{1}{2}} - \frac{d}{2}$ Then $n = \overline{\frac{2s}{d}+\frac{b^2}{4}}\vert^{\frac{1}{2}} - \frac{b}{2}\left(\frac{2a-d}{d}=b\right)$ and $n = \frac{b}{2} \pm \overline{\frac{b^2}{4}-\frac{2s}{d}}\vert^{\frac{1}{2}}\left(\frac{2l+d}{d}=b\right)$

II. According as the given things in any of the preceding *Problems* are choſen or related to another, ſo will the *Problem* be poſſible or impoſſible: For any three Numbers taken at random will not make a poſſible *Problem* of each of them, (tho' of ſome it will); becauſe there are particular Relations, within certain Limits, which the given Numbers muſt have to one another in moſt of theſe *Problems*; ſo that they may be poſſible, *i. e.* ſo that the three given things may belong to the ſame Progreſſion. The Poſſibility or Impoſſibility will appear by applying the Rules; for if the given Numbers are inconſiſtent, one or more of the things ſought will be found impoſſible, by ſome Abſurdity that will appear in applying the given Numbers to one another according to the Rule.

But now if you require how to *invent* three Numbers conſiſtent with one another, to make *Data* for any of theſe *Problems*; it may be done either of theſe Ways; *viz.*

1. Take any two Numbers whatever for a, d, and any Integer greater than 1, for n; and by theſe three find l, s; and thus you have five things all belonging to one *Progreſſion*, out of which to chuſe any 3 for *Data* of a *Problem*. The Reaſon of this Rule is plainly thus: That from any Number a we may raiſe a Progreſſion with any Difference d to any Number of Terms n we pleaſe.

2. Or take any two Numbers for a, l, ſo that a do not exceed l; and any Integer greater than 1 for n, and by theſe find d, s. The Reaſon of this is, that betwixt any two Numbers

bers a, l, any Number of Arithmetical Means may be placed, as has been ſhewn in *Corol.* to *Prob.* 5.

3. Therefore if a, n, d, or a, n, l are the Terms to be invented, we can find them by themſelves; and if any 2 of theſe 3, (as a, n. a, d n, d. a, l. n, l. d, l.) with any other Term [except that one Caſe d, l, s] are to be invented, we can find them without finding all the 5; yet one of the two things not required muſt be found: for we muſt take either a, n, d, or a, n, l, and by them find the other Term to be invented. But if d, l and s are to be invented, we muſt find all the 5 by means of a, n, d, or a, n, l.

But again, it may be required to invent the three things to be given in each of theſe *Problems*, without the Invention of any of the other two; which by the Rules now given cannot be done, except when a, n, d, or a, n, l are to be given. For this you have Rules in what immediately follows, when it is poſſible to be done.

III In the preceding *Problems*, no leſs than three things are neceſſary to be known, to make each of them determinate to one certain Solution: But if we ſuppoſe only two of the five things to be given for finding the other three; then, of ſuch *Problems*, ſome will be indeterminate, and have an infinite Number of Solutions, *i. e.* we can find an endleſs Variety of Numbers for the three things ſought, which will all ſatisfy the *Problem*: Alſo of theſe indeterminate *Problems*, ſome will be abſolutely indeterminate as to ſome of the things ſought, ſo that any Number whatever may be aſſumed. But others of the things ſought [and in ſome of theſe *Problems* all the things ſought] muſt be taken within certain Limits, which nevertheleſs admit of an infinite Variety of Solutions. Again; Others of theſe *Problems*, wherein only two things are given, will be determinate to a certain Number of Solutions, according to the different Circumſtances and Relations of the two given things; for there will be one or more Solutions, as theſe Circumſtances differ.

Of all theſe Problems, *Indeterminate* or *Determinate*, there are juſt 10; becauſe there are juſt ſo many different Choices of two things to be found in five. Thus; the five things being a, l, d, n, s, the Choices of 2, to make *Data* of a Problem, are theſe; a, l. a, d. a, n. a, s l, d. l, n. l, s. d, n. d, s. n, s. Of which there are 6 that are *Indeterminate*, and 4 *Determinate*.

§. 3. *Containing* Problems *concerning* Arithmetical *Progreſſions, wherein two things only are given to find the other three.*

THAT I may deliver the Rules and Demonſtrations of the following *Problems* in the moſt ſimple and eaſy manner; and that you may underſtand them aright, take theſe few previous Explanations.

1. Tho' I have ſhewn in moſt of the following *Problems*, how to find, by means only of the two given things, any one of the three things ſought; you are not to underſtand it, as if all theſe three Rules were to be applied in the ſame Solution, *i. e.* as if three Numbers found, one by each of theſe Rules, might be taken for the Solution of the *Problem*; becauſe there being a Variety of Solutions for each of theſe three things, any one Solution for each of them will not make a Combination that can ſolve the *Problem*; for this plain Reaſon, *viz.* when any one right Number is taken for any one of the three unknown things, this with the two given things determine the other two things ſought, according to the preceding *Problems*, which have but one limited Solution; ſo that we cannot with any one Solution, for one of the things ſought, join any one of the Solutions for the other two things; theſe being now determin'd by the Solution which we have choſen for the former one, together with the two given things: therefore the particular Rules for the different things ſought, are to be underſtood only as Steps in ſo many different Methods of ſolving the ſame *Problem*; which are to be applied thus: By the two given things find any one of

 the

the unknown things, according to the Rule given for it; then take the thing now found, with the two given things; and by theſe three find the other two things ſought, by that one of the preceding *Problems* where theſe three things are given: But the Rules for theſe are ſet down along with the other.

2. As any three things taken at random could not make the preceding *Problems* poſſible, ſo neither here will any two Numbers make any of the following *Problems* poſſible; there being certain Limitations, in reſpect of one another, under which they muſt be taken in ſome *Problems*, tho' not in all. That we may not encumber the *Problems* with theſe things, I ſhall here explain theſe Limitations where they ought to be, and ſhew where there are none. Thus:

a has no Limitation, and may be any Number whatever, and even o.

l may be any Number whatever, if it's not leſs than a or d, nor greater than s; but has no Limitation with reſpect to n.

d may be any Number whatever not exceeding l or s, or it may alſo be o; but has no Limitation with reſpect to a or n.

n muſt be an Integer greater than 1; and tho', ſtrictly ſpeaking, there is no Progreſſion without three Terms, yet we ſhall here allow of two Terms as the ſmalleſt Progreſſion.

s may be any Number whatever, not leſs than a, or d, or l; but has no Limitation with reſpect to n.

The Reaſon of theſe things is obvious from the Nature of a Progreſſion, which may begin with o, or any Number, and proceed by any or no Difference; and may conſiſt of two only, or any other Number of Terms.

Now in all the following *Problems* it's ſuppos'd that the two given things are conſiſtent, according to theſe Directions; and for the Limitations expreſſed in the Rules for finding one of the things ſought, they do not only comprehend the Conditions now mention'd, but ſome of them contain more ſtrict ones, becauſe regard is to be had to the reſt of the five things. And in the Reaſon given for theſe Rules, the general Conditions now mention'd are frequently ſuppoſed; which therefore muſt be always in view, becauſe ſuch ſimple things need not be repeated, unleſs where there is any danger of Obſcurity.

3. As to the Method of inveſtigating the following particular Rules, I ſhall here give you a general Account of it, that I may ſave the repeating of the ſame things, as otherways would be neceſſary in the demonſtrating of each of them.

In the firſt place, they depend upon the preceding *Problems*, and are diſcovered thus: I take that one of theſe *Problems* in which are given the two given things of the preſent *Problem*, and that one of the three things ſought that I would now firſt find; then by the Rules of that *Problem* for finding the remaining two things, I diſcover what Limitations that one I would now find lies under with reſpect to the two given things, that theſe two remaining things may be poſſible; and then I conclude, that the thing I ſeek being taken within this Limitation, it belongs to the ſame Progreſſion with the two given things; and is therefore a true Solution. But more particularly: Suppoſe a, l are given, and I would find s; I go to the preceding *Prob.* 6. wherein a, l, s are given; and there the Rules for finding n, d are theſe, $n = \frac{2s}{a+l}$, and $d = \frac{l^2 - a^2}{2s - a - l}$. And here to make n poſſible, it's plain that $2s$ muſt be Multiple of $a+l$, becauſe n muſt be an Integer greater than 1; and the ſame Condition of s will make d poſſible; for ſince a does not exceed l, there is nothing to make it impoſſible, but $a+l$ being greater than $2s$; which it is not, if $2s$ be a Multiple of $a+l$. Therefore if s is taken ſo, as $2s$ be a Multiple of $a+l$, [or alſo if we take for s any Multiple of $a+l$; from whence certainly follows that $2s$ is a Multiple of $a+l$] any ſuch Number ſolves the Queſtion for s; that is, s ſo taken with a, l, belong to the ſame Progreſſion. For if they did not, n, d could not be found by their Means, according to ſuch Rules as have been diſcovered and demonſtrated upon that very Suppoſition, that all the five things do belong to the ſame Progreſſion. This

This manner of drawing the Conclusion you are to suppose in all the following Demonstrations, which I shall never again repeat; but only shew you how that the Number sought being taken according to the Rule, is consistent with the Possibility of the remaining two things to be found. In doing of which, I have frequent Use for this *Principle, viz.* If one Number is equal to, greater or lesser than another, any Multiple, or aliquot Part of the former is also equal to, greater or lesser than the like Multiple or aliquot Part of the other; which being so very simple, it will be obvious in the Places where I use it, and therefore I shall not again repeat it.

We come now to the Problems, *whereof the first six are* Indeterminate, *and the other four* Determinate; *and mind, that the* Problems *here referred to, are those of the preceding* §.

PROB. XII. *Given* a, l, *to find* n, d, s.

1. For n. Take any Integral Number greater than 1; you have the Reason of this in the preceding *Scholium* 1.

2. For d. Take any Number such, that $\frac{l-a}{d}$ be an Integer, *i. e.* take any aliquot Part of $l-a$. For by *Prob.* 6. $n=\frac{l-a}{d}+1$, and $s=\frac{al+dl+l^2-a^2}{2d}$. Which are both possible, as d is taken; because n being an Integer, $\frac{l-a}{d}$ must be so; and if we divide $l-a$ by any Integer, and call the Quote d, the same d dividing $l-a$ will return for a Quote the former Integer; therefore any Number d, which is an aliquot Part of $l-a$, makes $\frac{l-a}{d}$ an Integer. As for s, it requires only that l be not less than a, which is the general Condition.

3. For s, take any Multiple of $a+l$, or the half of any such Multiple. For by *Prob.* 8. $n=\frac{2s}{a+l}$, and $d=\frac{l^2-a^2}{2s-a-l}$; which are both evidently possible, as s is limited.

PROB. XIII. *Given* n, d, *to find* a, l, s.

1. For a, take any Number, or even 0. The Reason is shewn already.

2. For l, take any Number not less than $d\overline{n-1}$. For by *Prob.* 7. $a=l-d\overline{n-1}$, and $s=\frac{2ln-dn^2+dn}{2}$. The former contains the very Conditions of the Rule, and the other requires only that $2ln+dn$ be greater than dn^2, or $2l+d$ greater than dn. But if l is at least $=dn-d$, then is $2l+d=2dn-2d+d=2dn-d$; which is greater than dn, because n is at least 2.

3. For s, take any Number greater than $\frac{dn^2-dn}{2}$. For by *Prob.* 10. $a=\frac{2s-dn^2+dn}{2n}$, and $l=\frac{2s+dn^2-dn}{2n}$. Now if s is greater than $\frac{dn^2-dn}{2}$, then is $2s$ greater than dn^2-dn, and $2s+dn$ greater than dn^2; which evidently makes a and l possible.

PROB. XIV. *Given* a, n, *to find* l, d, s.

1. For l, take any Number not less than a; the Reason is shewn above.

2. For d, take any Number greater; the Reason is also shewn above.

3. For

3. For s, take any Number greater than an. For by *Prob.* 9. $l = \frac{2s-an}{n}$, and $d = \frac{2s-2an}{n^2-n}$. The Reaſon of the Limitation is plain.

Prob. XV. *Given* l, n, *to find* a, d, s.

1. For a, take o, or any Number not exceeding l. The Reaſon is ſhewn above.

2. For d, take any Number not exceeding $\frac{l}{n-1}$. For by *Prob.* 7. $a = l - d\overline{n-1}$, and $s = \frac{2ln+dn-dn^2}{2}$. In the former $d\overline{n-1}$ muſt not exceed l, nor conſequently muſt d exceed $\frac{l}{n-1}$. In the other, $2ln+dn$ muſt be greater than dn^2; hence $2l$ greater than dn^2-dn, or $\overline{dn-1}\times n$; which is true, if d does not exceed $\frac{l}{n-1}$; for then $d\overline{n-1}$ does not exceed l, and conſequently $d\overline{n-1}\times n$ does not exceed ln; wherefore, laſtly, $2ln$ is greater than $d\overline{n-1}\times n$, as was required.

3. For s, take any Number leſs than $l\,n$, but not leſs than $\frac{ln}{2}$; for by *Probl.* IX $d = \frac{2ln-2s}{n^2-n}$, and $a = \frac{2s-ln}{n}$: The former requires only that s be leſs than ln, and the other that $2s$ be not leſs than $l\,n$, or s not leſs than $\frac{ln}{2}$.

Probl. XVI. *Given* a, d, *to find* n, l, s.

1. For n, take any Integer greater than 1.

2. For l, take any Number ſo that $\frac{l-a}{d}$ be integer; *i. e.* take any Multiple of d, and add a to it, the Sum is l; for by *Probl.* VI. $n = \frac{l-a}{d}+1$, and $s = \frac{ad+ld+l^2-a^2}{2d}$ The Reaſon of the Limitation is plain.

3. For s. This cannot be found without firſt finding ſome of the former; becauſe *Probl.* XI. whoſe *Data* are a, d, s, affords us nothing for this purpoſe.

Probl. XVII. *Given* n, s, *to find* a, l, d.

1. For a, take o, or any Number leſs than $\frac{s}{n}$; for by *Probl.* IX. $l = \frac{2s-an}{n}$ and $d = \frac{2s-2an}{n^2-n}$, which require only that $2an$ be leſs than $2s$, or an leſs than s, and, laſtly, n leſs than $\frac{s}{n}$.

2. For l, take any Number greater than $\frac{s}{n}$, but not exceeding $\frac{2s}{n}$; for by *Probl.* IX. $d = \frac{2ln-2s}{n^2-n}$, and $a = \frac{2s-ln}{n}$; the former requires that $l\,n$ be greater than s, or l greater than $\frac{s}{n}$; the other that ln do not exceed $2s$, or l not exceed $\frac{2s}{n}$.

3. For

3. For d, Take any Number less than $\frac{2s}{n^2-n}$; for by *Prob.* X. $l=\frac{2s+dn^2-dn}{2n}$, and $a=\frac{2s-dn^2+dn}{2n}$, the last of which puts the narrowest Limits upon d, *viz.* that $2s+dn$ be greater than dn^2, or $2s$ greater than dn^2-dn, and consequently s greater than $\frac{dn^2-dn}{2}$; which is the Condition of the Rule.

SCHOLIUM. This *Problem* is in effect the same as this, *viz.* To divide a certain given Number (s) into a given Number of Parts (n), such, that these Parts, from the least to the greatest, make a Progression *Arithmetical.*

PROBL. XVIII. *Given* a, s, *to find* l, n, d.

1. For l, Take any aliquot Part of $2s$, *i. e.* divide $2s$ by any Integer less than itself, so that the Quote be greater than a, and that when a is taken out of it the Remainder may not be less than a; that Remainder may be taken for l; and in order to this Solution begin with 2, and try all the Integers from that upwards, till you come to one which answers the Rule. The *Reason* is, because by *Probl.* VIII. $n=\frac{2s}{a+l}$, and $d=\frac{l^2-a^2}{2s-a-l}$, of which the former plainly requires the Limitation of the Rule, and the other is evidently possible upon the same Conditions. And lastly, *observe*, That there can be no more Solutions in this Method than there are Integers less than $2s$, which satisfy the Rule.

2. For n, Take any such Integer that an be less than s, *i.e.* take any Integer greater than 1, but less than $\frac{s}{a}$; for by *Probl.* IX. $l=\frac{2s-an}{n}$, and $d=\frac{2s-2an}{n^2-n}$; which require only that $2an$ be less than $2s$, or an less than s, or n less than $\frac{s}{a}$. And *observe*, That here the Number of Solutions are determined to the Number of Integers that are less than $\frac{s}{a}$, and greater than 1.

3. For d, It cannot be found till some of the other two, l or n, is found.

PROBL. XIX. *Given* l, s, *to find* a, n, d.

1. For a, Take any aliquot Part of $2s$, *i.e.* divide $2s$, by any Integer greater than 1, but less than itself, and such also that the Quote be greater than l, and that when l is taken out of the Quote, the Remainder do not exceed l; that Remainder may be taken for a. The *Reason* of these Limitations is, that $n=\frac{2s}{a+l}$, and $d=\frac{l^2-a^2}{2s-a-l}$ (*Probl.* VIII.) and the Solutions are limited to the Number of Integers that satisfy the Rule, which it's plain cannot be infinite.

2. For n, Take any Integer greater than 1, and such that n be greater than $\frac{s}{l}$, but less than $\frac{2s}{l}$, because, by *Probl* IX. $a=\frac{2s-ln}{n}$, and $d=\frac{2ln-2s}{n^2-n}$: The first requires only that $2s$ be greater than ln, and consequently $\frac{2s}{l}$ greater than n; the second requires only that ln be greater than s, and consequently n greater than $\frac{s}{l}$, so that the Number of Solutions is as many as the Integers greater than $\frac{s}{l}$ and less than $\frac{2s}{l}$.

3. For d, We cannot find it till a or n are first found.

PROBL. XX.

Prob. XX. *Given* d, l, *to find* a, n, s.

Here s cannot be found, till a or n is found; and these may be both found together in one Operation. Thus: Take any Number not exceeding l, and such also that it be equal to, or some Multiple of d, (by trying all the Multiples of d from $2d$, which will not exceed l); then is $n-1=1$, if the Number assumed is $=d$: But if it's a Multiple of d, the Multiplier is $=n-1$; and the Remainder, after the assumed Number is taken from l, is $=a$.

The Reason is, Because by *Prob.* 7. $a=l-d\times\overline{n-1}$; where if $d\times\overline{n-1}=d$, then is $n-1=1$; otherwise, if $d\times n-1=md$, then $n-1=m$.

Prob. XXI. *Given* d, s, *to find* n, a, l.

Here neither a nor l can be found till n be known; which may be found thus: Take any Integer greater than 1, and such also that $nn-n$ be less than $\frac{2s}{d}$; for by *Prob.* 10. $a=\frac{2s+dn-dnn}{2n}$, and $l=\frac{2s+dnn-dn}{2n}$. By the first it follows, that dnn must be less than $2s+dn$; and taking dn from both, it follows that $dnn-dn$ must be less $2s$; and lastly, (dividing both by d) that $nn-n$ must be less than $\frac{2s}{d}$. The other Part l is evidently possible, with the same Limitation.

General Corollary.

It's now manifest, how by these last ten *Problems* we can invent any three things that shall make any of the former seven *Problems* possible; and that after various manners, by taking any two of the three, under the general Conditions; and then with these two finding the other one, by that one of these last *Problems*, wherein these two things are given.

CHAP. III.

Of Geometrical *Proportion.*

§. 1. *Containing the more general Doctrine common to both* Conjunct *and* Disjunct *Proportionals.*

Observe, *In all that follows, I mark these Words,* Geometrical Proportion, *and* Geometrically Proportional, *by this* :: l; *and the Words* Continued Geometrical Proportion *by this* ∺ l.

Again; *When any Axioms are cited, you are to understand the* Axioms *at the End of* Chap. 1. *of this Book; and citing any of the general Corollaries there also explained, I mark them thus,* g. Cor.

PROBLEM I.

Having three Numbers given, to find a fourth, :: l.

RULE 1. FIND the Ratio of the first and second Terms, either

1. By dividing the greater Term by the lesser; and if the Antecedent is the lesser, multiply; or if the greater, divide the third Term by that Ratio; the Product or Quote is the 4th sought. Or,

2. Take the Quote of the first Term, divided by the second, for the Ratio; and by it divide, or by its Reciprocal (*i. e.* the Quote of the second Term divided by the first) multiply the third Term; the Quote or Product is the fourth sought.

Exam. 1. To these 2 : 6 :: 5, a 4th is 15. For the Ratio of 2, 6 is 3, by which 5 multiply'd produces 15.

Exam. 2. To these 24 : 20 :: 6, a 4th is 5. For the Ratio of 24 to 20 is $1\frac{4}{20} = 1\frac{1}{5} = \frac{6}{5}$, and $6 \div \frac{6}{5} = \frac{30}{6} = 5$.

DEMON. The Reason of this Rule is plainly contained in the Definitions, and needs no farther Explication: or you may see it particularly in the *Coroll.* to the Definition of Geometrical Relation, which does in effect contain this *Problem*; for there it is shewn how to find a Number in any given Ratio to a given Number; and here the first and second Terms contain the Ratio, in which the 4th Term sought ought to be to the 3d. Or we shall set this once more before us in this universal Representation of 4 Numbers :: *l*, viz. A : A*r* :: B : B*r*; wherein *r* being a whole or mixt Number, is the Ratio of A to A*r* in the one View, and its Reciprocal in the other; and it is manifest, that B*r* is the fourth sought to these A : A*r* :: B; also A is a fourth to these B*r* : B :: A*r*, according to the Method of the foregoing Rule; which is therefore good.

RULE 2. Multiply the second and third Terms together, and divide the Products by the first, the Quote is the fourth sought.

So in *Exam.* 1. $15 = \frac{6 \times 5}{2} = \frac{30}{2}$; and in *Exam.* 2. $5 = \frac{20 \times 6}{24}$.

Take this other *Example*: To these $4:5::7$, the fourth is $8\frac{3}{4}=\frac{7\times 5}{4}$.

Universally: To these 3, $A:B::C$, a 4th is $\frac{BC}{A}$.

DEMON. The Reason of this Rule will easily appear from the preceding. For let the Ratio of $A:B$ be taken $\frac{A}{B}$, the Reciprocal of it is $\frac{B}{A}$; by which the 3d Term C being multiplied, the Product is the fourth sought by the preceding Rule: But this Product, according to the Rule of multiplying Fractions, is $\frac{BC}{A}$, if A, B, C are all Integers. But tho' they are not all Integers, yet it has been shewn in *Schol.* after *g. Cor.* 20. that the Quote of any two Numbers, B, A, is multiplied or divided by taking the Divisor and Dividend as the Numerator and Denominator of a Fraction, and applying the Rules for Fractions; therefore the Quote of $B\div A$ taken thus, $\frac{B}{A}$; and multiplied by C, produces this Expression, $\frac{BC}{A}$; which is according to the Rule, *viz.* $BC\div A$.

Or the Truth of this Rule we may see also in this Representation, $A:Ar::B:Br$; where it is plain, that $ArB\div A=Br$, and $ArB\div Br=A$. For if any Number is first multiplied by another, and the Product divided by the Multiplier, the Quote is necessarily the Number multiplied; which is evidently the Case here: for $Ar\times B$ is $=A\times rB$, and $A\times rB\div A=rB$.

COROL. Having two Numbers given, we may find a 3d by dividing the Square of the second Term by the first. Thus: To $2:6$, a third is $18=\frac{6\times 6}{2}$. For since $2:6::6:18$, then 18 is a fourth to $2:6::6$; which reduces this Case to the preceding.

THEOREM I.

IF four Numbers are $::l$, the Product of the two Extremes is equal to the Product of the two Means. And *reversly*; if these Products are equal, the four Numbers are $::l$.

Thus; If $A:B::C:D$, then $AD=BC$. In Numbers, $2:3::4:6$, and $2\times 6=3\times 4=12$.

DEMON. By the preceding *Problem*, $D=\frac{BC}{A}$; hence $DA=BC$.

Or thus: Since $A:B::C:D$, these Quotes are equal, *viz.* $\frac{A}{B}=\frac{C}{D}$; then the Products of the Divisor of each by the Dividend of the other are equal, (see *general Scholium* at the End of *Chap* 1.) that is, $AD=BC$.

Or thus: $AD:BD::A:B$, (*g. Cor.* 15.) also $BC:BD::C:D$. But $A:B::C:D$; therefore $AD:BD::BC:BD$, (*Ax.* 3.); hence $AD=BC$, (*Ax.* 1.)

Or lastly, Let $4::ls$ be thus represented, $A:Ar::B:Br$; then it is manifest that $ABr=ArB$.

For the *Reverse*: If $AD=BC$, then is $A:B::C:D$. For, by equal Division, $A=\frac{BC}{D}$; and again, $\frac{A}{B}=\frac{C}{D}$; hence $A:B::C:D$.

Or thus: $AD:BD::A:B$, and $BC:BD:C:D$; and since $AD=BC$, therefore $BC:BD::A:B$; hence $A:B::C:D$, (*Ax.* 3.)

Or

Or alſo thus: A 4th :: l to A, B, C is poſſible; ſuppoſe that to be N: *i. e.* A : B :: C : N; then is $AN = BC$ (by the *Theor.*); but $AD = BC$ (by Suppoſition); therefore $AN = AD$: Hence $N = D$; therefore D is the 4th :: l, that is, A : B :: C : D.

COROLLARIES.

1. If three Numbers are $\div l$, the Product of the Extremes is equal to the Square of the middle Term. Thus; A : B : C being $\div l$, $AC = B^2$. In Numbers, 2 : 4 : 8 are $\div l$, and $2 \times 8 = 4 \times 4$.

The *Reverſe* of this *Coroll.* is alſo true, *viz.* that if $AC = B^2$, then is A : B :: B : C.

2. If four Numbers are :: l, A : B :: C : D, the Product of all the four Terms is a ſquare Number, whoſe Root is the Product of the Extremes, or of the Means; theſe Products being equal. So $AD = BC$. Hence $AD \times BC = AD^2 = BC^2$.

Again; Three Numbers $\div l$ multiplied all together produce a Cube Number, whoſe Root is the middle Term. Thus: A : B : C being $\div l$, then $AC = B^2$, and $B^2 \times B = AC \times B = B^3$.

SCHOLIUMS.

1. As this *Theorem* is demonſtrated without the *Prob.* 1. ſo the 2d Rule of that *Problem* is an evident Conſequence of this *Theorem*: For, if A : B :: C : D; then $D = \frac{BC}{A}$, becauſe $DA = BC$.

2. All the general Corollaries relating to :: l, in the End of *Chap.* 1. may be moſt eaſily demonſtrated by this *Theorem*. For in all the Proportions there ſtated, we ſhall find this certain Mark of :: l, *viz.* the equal Products of the Extremes and Means.

3. The Reverſe of this *Theorem* may be put in this Form, *viz.* If the Product of any two Numbers is equal to the Product of other two, theſe 4 are reciprocally :: l. Thus: If $AD = BC$, then A : B :: C : D. And here obſerve, that the Factors of theſe equal Products are ſaid to be reciprocally :: l; becauſe one of the two comparative Terms is taken out of the one Product, and the other out of the other Product.

PROBLEM II.

Of four Numbers :: l, having the two Extremes and one Mean, to find the other Mean; or to find one Mean in $\div l$ betwixt two Numbers.

RULE 1. Divide the Product of the Extremes by the known Mean, the Quote is the other.

DEMONSTR. This follows from the laſt *Theorem*; for if A : B :: C : D, then is $AD = BC$; and hence $AD \div B = C$, and $AD \div C = B$.

RULE 2. For one Geometrical Mean betwixt two Numbers, take the Square Root of the Product of the two Extremes.

DEMON. If A : B : C are $\div l$, then $AC = B^2$; hence $\overline{AC}^{\frac{1}{2}} = B$. (*Ax.* 1. *B.* III. *C.* I.)

Example: Betwixt 2 and 8 a Geometrical Mean is 4; for $2 \times 8 = 16$, and the Square Root of 16 is 4.

But if the Product of the Extremes has not a determinate Square Root, the Mean ſought is a Surd, or an infinite Series of decreaſing Quantities, as has been explain'd in *Book* III. So betwixt 2 and 7 the Mean is the Square Root of $14 = 3.74$, *&c.* which is continued *in infinitum*, according to the Method of Approximation, explained in *Book* III. *Ch.* I.

THEOREM II.

IF four Numbers are :: *l*, A : B :: C : D, they are so also *reversely*; that is, making the Consequents the Antecedents: Thus, B : A :: D : C, or D : C :: B : A.

In Numbers, If it be 2 : 3 :: 4 : 6, then it is 3 : 2 :: 6 : 4.

DEMONSTR. This follows evidently from the *Definition*; for since A and C do equally contain, or are contained in B and D, then, reversely, B and D are equally contained in, or do equally contain, A and C; which is the nature of :: *l*.

Or thus: Since $\frac{A}{B} = \frac{C}{D}$, then the reciprocal Quotes are equal (see the general *Schol.* at the End of *Chap.* I.) *viz.* $\frac{B}{A} = \frac{D}{C}$, *i.e.* B : A :: D : C.

Or also thus: $\frac{B}{B} = \frac{D}{D}$, and $\frac{A}{B} = \frac{C}{D}$; therefore, (by *Ax.* 1.) $\frac{B}{B} : \frac{A}{B} :: \frac{D}{D} : \frac{C}{D}$. Again; $\frac{B}{B} : \frac{A}{B} ::$ B : A, and $\frac{D}{D} : \frac{C}{D} ::$ D : C (general *Coroll.* 15.); therefore B : A :: D : C (*Ax.* 3.)

Or the same Truth appears simply in this Representation, *viz.* A : A *r* :: B : B *r*; whence A *r* : A :: B *r* : B, the Ratio being still the same.

Or lastly, It follows from the equal Product of the Extremes and Means; for all the Change made by reversing the Terms is, that the Extremes are become the Means, and the Means become the Extremes, and the Product of the Extremes and Means are still equal, which makes :: *l*.

THEOREM III.

IF four Numbers are :: *l*, A : B :: C : D, they are so also *alternately*; that is, comparing the two Antecedents to one another, and the two Consequents: Thus, A : C :: B : D.

In Numbers, if 3 : 5 :: 6 : 10, then 3 : 6 :: 5 : 10.

DEMONSTR. These Quotes being equal, $\frac{A}{B} = \frac{C}{D}$, the alternate Quotes are also equal, $\frac{A}{C} = \frac{B}{D}$ (see general *Schol.* at the End of *Chap.* 1.); hence A : C :: B : D.

Or thus: A and C are the same Fractions, proper or improper, of B, D; but the same Fractions of two Numbers are the same Fractions of one another as these Numbers are (*Coroll.* 6. *Lemma* 2. *Chap.* I. *Book* II.), *i.e.* A is the same Fraction of C as B is of D, and Like or Equal Fractions make equal Ratios: Therefore A : C :: B : D.

Or we may reason thus, $\frac{A}{B} : \frac{C}{B} ::$ A : C (gen. *Coroll.* 15.); also $\frac{C}{D} : \frac{C}{B} ::$ B : D (gen. *Coroll.* 16.); but $\frac{A}{B} = \frac{C}{D}$, therefore $\frac{A}{B} : \frac{C}{B} ::$ B : D; hence, lastly, A : C :: B : D.

Or represent four Numbers :: *l* thus, A : A *r* :: B : B *r*; then it is plain, that A : B :: A *r* : B *r* (by gen. *Coroll.* 15.)

Or, lastly, This does also follow from the equal Product of Extremes and Means, whose Factors are not changed, except in their Order, which does not alter the Product, it being still A D = B C.

COROLLARIES.

1. Of four Numbers :: *l*, A : B :: C : D, if A, B, are lesser than C, D, or these lesser than those, the two lesser are Like Fractions of the two greater; because A : C :: B : D; or, reversly, C : A :: D : B.

2. If

2. If three Nnmbers are given, to find a 4th :: *l*, and if the firſt is an aliquot Part or Multiple of the third, and that this can be eaſily diſcerned, then the 4th will be more eaſily found by making the 3d Term the 2d, and applying the firſt *Rule* of *Probl.* I.

Example: To theſe, $4:7::12$, a 4th is 21; for 4 is the third Part of 12, therefore I multiply 7 by 3.

3. If 4 Numbers are :: *l*, $A:B::C:D$, then if A is leſſer, greater, or equal to C, B is alſo greater, leſſer, or equal to D, becauſe $A:C::B:D$.

THEOREM IV.

If four Numbers are :: *l*, $A:B::C:D$, they are ſo alſo *compoundly*; i. e. the Sums of the Antecedents and of the Conſequents are proportional with each Antecedent and its Conſequent: Thus, $A+C:B+D::A:B$, or $::C:D$. Alſo the Sums of each Antecedent and its Conſequent are proportional with the two Antecedents, or the two Conſequents. Thus $A+B:C+D::A:C$, or $::B:D$.

Example: $3:5::6:10$, and $9:15::3:5$, alſo $8:16::3:6$.

DEMONSTR. The Antecedents A, C, are Like Fractions (proper or improper) of their Conſequents, B, D (by the Nature of Ratios); but the Sum of the Like Fractions of two Numbers is the Like Fraction of the Sum of theſe Numbers, (*Lem.* 2. *Chap.* I. *Book* 2.) that is, $A+C$ is the ſame Fraction of $B+D$, as A is of B, or alſo C of D; and the ſame Fraction is the ſame Ratio; therefore $A+C:B+D::A:B::C:D$. For the ſecond Part, *viz.* $A+B:C+D::A:C::B:D$, this follows from the ſame Principle; having firſt alternated the given Numbers, thus, $A:C::B:D$.

But we may demonſtrate this otherwiſe, thus, $\frac{A}{B}=\frac{C}{D}$, and $\frac{B}{B}=\frac{D}{D}$; hence $\frac{A}{B}+\frac{B}{B}=\frac{C}{D}+\frac{D}{D}$; and by the Addition of theſe Quotes, conſidered as if they were Fractions (ſee gen. *Schol.* at the End of *Ch.* I.) it is $\frac{A+B}{B}=\frac{C+D}{D}$; hence $A+B:C+D::B:D$, which is the one Part; and for the other, ſince $\frac{A}{C}=\frac{B}{D}$, and $\frac{C}{C}=\frac{D}{D}$; hence $\frac{A+C}{C}=\frac{B+D}{D}$, and $A+C:B+D::C:D$, or $::A:B$.

Or alſo from the equal Products of Extremes and Means, $\overline{A+C}\times B=\overline{B+D}\times A$; for $\overline{A+C}\times B=AB+CB$, and $\overline{B+D}\times A=AB+AD$ (either by *Lem.* 3. *C.* V. *B.* I. if B and A are Integers; or by *Lem.* 2. *Ch.* I. *Book* II. if they are Fractions): But $AD=BC$ (*Theor.* 1.) hence $AB+CB=AB+AD$; *i. e.* $\overline{A+C}\times B=\overline{B+D}\times A$; wherefore $A+C:B+D::A:B$ (by the Reverſe of *Theor.* 1.)

Or ſee it in this Repreſentation, $A:Ar::B:Br$; hence it is plain that $A+B:Ar+Br\ (=\overline{A+B}\times r)::A:Ar$, the common Ratio being *r*; or $A+Ar\ (=\overline{1+r}\times A)$. $B+Br\ (=\overline{1+r}\times B)::A:B$; theſe being equally multiplied in the firſt Pair by $\overline{1+r}$: Or in both Caſes you ſee an equal Product of Extremes and Means.

COROLLARIES.

1. If there are ever ſo many Couplets of Numbers in the ſame Ratio, the Sum of any Number or all of the Antecedents is to the Sum of the ſame Number of the Conſequents, as any of the Antecedents to its Conſequent. Thus, if $a:A::b:B::c:C::d:D$, *&c.* then $a+b+c+d:A+B+C+D::a:A::$ *&c.* For $a+b:A+B::a:A$, or $c:C$, or $d:D$; hence again, $a+b+c:A+B+C::c:C$, or $d:D$; and again, $a+b+c$

$+d : A+B+C+D :: d : D$. Or it follows simply from the same general Principles as the *Theorem*, viz. The Sum of the Like Fractions of any Numbers, however many there be of them, is the Like Fraction of the Sum of these Numbers.

2. If four Numbers $::l$ are equally multiplied, the Products are also $::l$, and in the same Ratio: Thus, if $A:B::C:D$ are multiplied by r, these are $::l$, $Ar:Br::Cr:Dr$; for Multiplying is only a repeated Addition: But this follows also from gen. *Coroll* 15. for $Ar:Br::A:B$, and $Cr:Dr::C:D$; hence $Ar:Br::Cr:Dr$.

3. If $A:B::C:D$, and $M:N::O:P$; and if the Ratio is the same in both Ranks, *i e.* $A:B::M:N$, and $C:D::O:P$, then the Sums of their corresponding Terms is also a Rank of $::l$'s; that is, $A+M:B+N::C+O:D+P$.

Observe, The first *Coroll.* is the same thing in effect as this Proposition, *viz.* if two or more Numbers are composed by addition of the same number of Parts, those of the lesser Whole being lesser, compared one to one respectively from the least to the greatest, than those of the greater Whole, and all in the same Ratio to their Correspondents in the other, These *Wholes* are also in the same *Ratio.*

Theorem V.

If four Numbers are $::l$, $A:B::C:D$, they are so *divisively*: Thus, the Difference of the Antecedents is to the Difference of the Consequents as each Antecedent to its Consequent, $A-C:B-D::A:B$, and also as $C:D$. Also, the Differences of each Antecedent and its Consequent are as each Antecedent or Consequent to the other, $A-B:C-D::A:C$, and also as $B:D$; and it is the same thing if the Antecedents are lesser than their Consequents, or A lesser than C, and B than D; for then it is $C-A:D-B::A:B$, and $B-A:D-C::A:C$.

Example: $3:7::15:35$, and $12:28::3:7$; also $4:20::3:15$.

Demonstr. A, C, are Like Fractions of B, D, and (by *Lem.* 3. *Ch.* I. *Book* II.) the Difference of the Like Fractions of two Numbers is the Like Fraction of the Difference of these Numbers; that is, $A-C$ or $C-A$ is the same Fraction of $B-D$ or $D-B$, as is A of B or C of D; therefore $A-C$ or $C-A:B-D$ or $D-B::A:B::C:D$. The second Part, *viz.* $A-B:C-D::A:C::B:D$, follows from the Alternation of the given Numbers.

Or we may demonstrate this *Theorem* thus: $\frac{A}{C}=\frac{B}{D}$, and $\frac{C}{C}=\frac{D}{D}$; hence $\frac{A-C}{C}=\frac{B-D}{D}$; that is, $A-C:B-D::C:D$, or $A:B$.

Or thus also; $\overline{A-C}\times B=AB-CB$, and $\overline{B-D}\times A=AB-AD$; but $AD=BC$, hence $AB-CB=AB-AD$; that is, $\overline{A-C}\times B=\overline{B-D}\times A$, the Product of the Extremes equal to that of the Means. Hence $A-C:B-D::A:B$.

Or thus; $A:Ar:B:Br$, and $A-B:Ar-Br$ $(=\overline{A-B}\times r)$ $::A:Ar$, the common Ratio being r. Also $Ar-A$ $(=\overline{r-1}\times A)$ $:Br-B$ $(=\overline{r-1}\times B)$ $::A:B$, these being equi-multiplied in the first Pair by $\overline{r-1}$; or in both Cases you see an equal Product of Extremes and Means.

Scholium. This *Theorem* is the same in effect as this Proposition, *viz.* If any two Numbers are in the same Ratio to one another as the Parts taken away, the Parts remaining are also in the same Ratio. Thus, suppose A, B to be greater than C, D, then A, B, being consider'd as two Wholes, C, D are the Parts taken away, and $A-C$, $B-D$, are the

the Parts remaining. And the Propoſition is clearly expreſs'd in Signs thus, $A - C : B - D :: A : B :: C : D$; and if C, D are the greater, it is $C - A : D - B :: C : D :: A : B$.

We may alſo expreſs the Propoſition thus: If two Numbers are each the Sum of other two Numbers, or Parts, and if the one Whole is in the ſame Ratio to the other as one Part of the firſt Whole is to one Part of the other, then the other Parts are alſo in the ſame Ratio: Thus, A, B being the Wholes, C, and $A - C$ are the Parts of A, and D, $B - D$, the Parts of B; for $A - C + C = A$, alſo $B - D + D = B$.

The like *Corollaries* follow from this *Theorem* as from the laſt, by applying *Subtraction* and *Diviſion*, as we did there *Addition* and *Multiplication*: To which we may add the following

COROLLARIES.

1. Of four Numbers, A, B, C, D, if it is $A + C : B + D :: A : B$, or as $C : D$; but let it not be affirm'd to be both as $A : B$ and alſo as $C : D$, yet this will follow, that $A : B :: C : D$, and conſequently that $A + C : B + D$ is both as $A . B$ and $C : D$; For $A + C$ and $B + D$ are two Wholes, which being ſuppoſed in the ſame Ratio as any one of the Parts, $A : B$, the other two Parts are alſo in the ſame Ratio, by this *Theorem* and *Scholium*; that is, $A : B :: C : D$, and hence alſo $A + C : B + D :: A : B :: C : D$. From this again follows, that

2. If two Wholes are compoſed each of two Parts, and if the Parts of the one are not both in the ſame Ratio to the Parts of the other, the Wholes are in neither of theſe Ratios; for if they were in the Ratio of any one of them, they would be in the Ratio of both, and conſequently the Parts would be in the ſame Ratio, contrary to Suppoſition.

3. Of four Numbers, A, B, C, D, if $A - C : B - D :: A : B$, or $C : D$; but we don't ſay *and alſo as* $C : D$, then it will be $A : B :: C : D$, and conſequently $A - C : B - D :: A : B :: C : D$, for A, B are two Wholes, C and $A - C$ are the two Parts of A; and D, $B - D$ the Parts of B; but one of the Parts being in the Ratio of the Wholes, *viz.* $A - C : B - D :: A . B$, then are the other Parts, by this *Theorem*; that is, $A : B :: C : D$; and if $A - C : B - D :: C : D$; that is, if both the Parts of the one Whole are in the ſame Ratio to thoſe of the other, the Wholes are in the ſame Ratio, by *Theor.* IV. that is, $A : B :: C : D$.

THEOREM VI.

If four Numbers are $:: l$, $A : B :: C : D$, they are ſo alſo *mixtly*; that is, comparing the Sums and Differences of the Antecedents and Conſequents; thus,

$$A + C : B + D :: A - C, \text{ or } C - A : B - D, \text{ or } D - B.$$
$$\text{Alſo } A + B : C + D :: A - B, \text{ or } B - A : C - D, \text{ or } D - C.$$

DEMONSTR. This follows from the two laſt; for the Sums or Differences here compared are in the ſame Ratio of one of the Antecedents to its Conſequent, *viz.* as $A : B$ in the firſt Part; or as the one Antecedent to the other, *viz.* $A : C$, in the ſecond Part.

SCHOLIUM. The three laſt *Theorems* may alſo be taken *reverſly* or *alternately*.

THEOREM VII.

If there are ever ſo many Ranks of four Numbers $:: l$, and if any two of the comparative Terms (*i.e* the 1ſt and 2d, or 1ſt and 3d, or 3d and 4th, or 2d and 4th) are common to all the Ranks, then the other Couplets in every Rank are all in the ſame Ratio. Or if it's thus, *viz.* Two comparative Terms of the 1ſt Rank common to the 2d Rank,

2d Rank, and the remaining two of the 2d Rank common to the 3d Rank, and ſo on, then the other Couplets in each Rank will be $::l$.

DEMONSTR. All this is the ſimple and immediate Application of *Axiom* 3. and *Theorem* III. and needs no more Explication but a few *Examples*, where you may ſee the different Forms in which theſe things may appear.

EXAMPLES.

(1.)	(3.)	(5.)
If $A:B::C:D$, and $C:D::E:F$; then $A:B::E:F$.	If $A:B::C:D$, and $E:B::F:D$; then $A:E::C:F$.	If $A:B::C:D$, and $C:D::E:F$, and $E:F::G:H$; then $A:B::E:F::G:H$.
(2.)	(4.)	(6.)
If $A:B::C:D$, and $B:D::E:F$; then $A:C::E:F$.	If $A:B::C:D$, and $B:E::D:F$; then $A:C::E:F$.	If $A:B::C:D$, and $A:C::E:F$, and $G:H::A:C$; then $B:D::E:F::G:H$.

THEOREM VIII.

IF there are two Ranks of 4 Numbers $::l$, which have two comparative Terms common to both, the Sums or Differences of the Antecedents and Conſequents of the two different Couplets are in the ſame Ratio with the Antecedent and Conſequent of the common Couplet. Thus:

If $A : B::C : D$,
and $E : B::F : D$;
then $A+E:B::C+F:D$;
alſo $A-E:B::C-F:D$.

DEMON. By the preceding $A:C::E:F$; whence $A+E$, or $A-E:C+F$, or $C-F::B:D$; or alternately, as in the Margin.

Again; Becauſe $A+B:C+D::B:D$, and $E+B:F+D::B:D$; therefore theſe Proportions are alſo true, *viz.*

$A+E:C+F::A+B:C+D::E+B:F+B$,
and $A-E:C-F::A-B:C-D::E-B:F-B$.

Alſo any Couplet of the firſt Rank is $::l$ with any of the ſecond; being all as $B:D$.

Again; It is alſo $A+E:B+B::C+F:D+D$. Since $2B:2D::B:D$.

THEOREM IX.

IF there are two Ranks of four Numbers $::l$, whereof the Extremes or Means of the one are the ſame as the Extremes or Means of the other; or if they are reverſly the Means or Extremes of the other; then the remaining four Terms are reciprocally $::l$. *i. e.* make the remaining two of the one Rank the Extremes, and thoſe of the other the Means, and theſe four are $::l$. Thus:

If $A:B::C:D$, and $E:B::C:F$; then $A:E::F:D$.	If $A:B::C:D$, and $A:E::F:D$; then $B:E::F:C$.	If $A:B::C:D$, and $B:E::F:C$; then $A:E::F:D$.	If $A:B::C:D$, and $E:A::D:F$; then $B:E::F:C$.

DEMON. From the equal Products of Extremes and Means, it is $AD=BC=EF$; in which all theſe Concluſions are comprehended.

THEO-

THEOREM X.

IF four Numbers are :: *l*, and if any two of the comparative Terms are equally multiplied or divided; or if the one Extreme or Mean is multiplied, and the other equally divided: or again, if the one Extreme is multiplied, and the other equally divided; and at the ſame time the one Mean multiplied by any other, or the ſame Number, and the other equally divided; the Proportionality ſtill remains, tho' in the ſecond Caſe the Ratio is changed; and will be alſo in the third Caſe, when two different Multipliers are employed. Thus: If A : B :: C : D, then theſe Proportions follow, *viz.*

$$A : B :: Cn : Dn. \qquad An : B :: Cn : D.$$

$$A : B :: \frac{C}{n} : \frac{D}{n}. \qquad A : \frac{B}{n} :: C : \frac{D}{n}.$$

$$An : B :: C : \frac{D}{n}. \qquad A : \frac{B}{n} :: Cn : D.$$

$$An : Bn :: Cn : Dn. \qquad \frac{A}{n} : \frac{B}{n} :: \frac{C}{n} : \frac{D}{n}.$$

$$An : Br :: Cn : Dr. \qquad \frac{A}{n} : \frac{B}{r} :: \frac{C}{n} : \frac{D}{r}.$$

$$\frac{A}{n} : \frac{B}{n} :: \frac{C}{r} : \frac{D}{r}. \qquad An : Br :: \frac{C}{r} : \frac{D}{n}.$$

$$\frac{A}{n} : Br :: \frac{C}{n} : Dr.$$

$$An : \frac{B}{r} :: Cr : \frac{D}{n}.$$

Again; Inſtead of dividing them, we may apply the given Numbers as Diviſors; which will make the following Proportions:

$$\frac{n}{A} : \frac{n}{B} :: \frac{n}{C} : \frac{n}{D}. \qquad \frac{n}{A} : \frac{n}{B} :: \frac{r}{C} : \frac{r}{D}.$$

$$\frac{n}{A} : \frac{n}{B} :: D : C. \qquad \frac{A}{n} : \frac{B}{r} :: \frac{r}{D} : \frac{n}{C}.$$

$$An : \frac{r}{D} :: Cn : \frac{r}{B}. \qquad \frac{r}{A} : \frac{n}{B} :: Dr : Cn.$$

DEMON. In all theſe Concluſions, and many more that may be contrived of this Nature, the Truth of the Proportion is evident from the equal Product of the Extremes and Means; founded all upon this, that $AD = BC$, $nr = nr$, $\frac{A}{B} = \frac{C}{D}$, and $\frac{r}{n} = \frac{r}{n}$.

You'll find alſo other complex ways of arguing with proportional Numbers in the next Chapter.

§. 2. *Of* Geometrical *Progreſſions.*

Obſerve, *By the Diſtance of one Term of a Series from another, is meant the Number of Terms from the one excluſive to the other incluſive; or including both, it is the Number of Terms leſs* 1. *So if the Number of Terms is* n, *the Diſtance of the Extremes is* n — 1.

PROBLEM III.

Having the firſt Term and Ratio to raiſe a Geometrical Series.

RULE. IF the given Ratio is a whole Number or mixt, then for an increaſing Series multiply, and for a decreaſing divide the firſt Term by the Ratio, the Product or Quote is the ſecond Term; which multiplied or divided by the Ratio, gives the third Term, and ſo on. But if the Ratio is a proper Fraction, this of itſelf determines that the Series ought to increaſe, and we muſt multiply by the Reciprocal of the Ratio.

Example 1. Firſt Term 2, Ratio 3, the increaſing Series is 2 : 6 : 18 : 54, *&c.* and the decreaſing Series $2 : \frac{2}{3} :: \frac{2}{9} : \frac{2}{27}$, *&c.*

Example 2. First Term 2, Ratio 4, the Series is $2 : 8 : 16 : 32$, *&c.* Or thus, $2 : \frac{1}{2} : \frac{1}{8} : \frac{1}{32}$, *&c.*

Demon. The Reason of this Rule is manifestly contained in the Definition of Geometrical Relation; see the *Coroll.* after it: for when the Ratio is a whole or mixt Number, which soever of the two Views there explained it is taken in, the Rule is good; because for an increasing Series, a whole or mixt Number is in the first View the Ratio, and in the second it is the reciprocal Ratio; and by the *Coroll.* referred to, the first Term or Antecedent ought to be multiplied by this; and for a decreasing Series, a whole or mixt Number is the Ratio in both Views, and Division is the Rule. Lastly, If the given Ratio is a proper Fraction, it is taken in the second View, and necessarily infers an increasing Series; and therefore the Antecedent multiplied by the reciprocal Ratio produces the Consequent.

SCHOLIUMS.

1. If the Ratio is expressed by two Numbers ordered in a certain Comparison, the one as Antecedent, and the other as Consequent, then, that determines whether the Series decreases or increases; and accordingly, if we make a Fraction of the Antecedent set over the Consequent, (which is the Ratio in the second View) and by it divide, we shall raise an increasing or decreasing Series, acccording as the Ratio is a proper or improper Fraction. For *Exam.* If the Ratio is thus expressed, *viz.* the Ratio of 2 to 3, or $2 : 3$; then dividing by $\frac{2}{3}$ makes an increasing Series. But if it's the Ratio of $3 : 2$, then dividing by $\frac{3}{2}$ makes a decreasing Series.

2. Again: If the two first Terms of a Series are given, (which do contain the Ratio) then the Series is continued by the common Rules of finding a third or fourth $\div l$, *viz.* finding a third to the two given ones, and then a fourth to the same two, and the Term last found. And so a Series from $a : b$, whether increasing or decreasing, will be thus represented, $a : b : \frac{b^2}{a} : \frac{b^3}{a^2} : \frac{b^4}{a^3}$, *&c.* But when a Number distinct from the two first Terms is given for the Ratio, then

3. Any Geometrical Progression, whether increasing or decreasing from a given Number a, may be clearly represented thus, $a : ar^2 : ar^3$, *&c.* adding still an Unit at every Step to the Index of r: for according as r is supposed greater or lesser than 1, so is the Series increasing or decreasing. If the Series increases, then is r the Ratio in one of the two Senses explained, or the Reciprocal of it in the other. For it is the Quote of ar, the greater Term divided by a the lesser; but the reciprocal Quote of a divided by ar, which is the Ratio in another View. Again; If the Series decreases, r is a proper Fraction, and is the Reciprocal of the Ratio taken in either View, which are in this Case the same, *viz.* the Quote of a divided by ar, which is $\frac{a}{ar} = \frac{1}{r}$, whose Reciprocal is r.

Wherefore, if we take the Ratio always under the Notion of the Quote of the Antecedent divided by the Consequent; then r is to be understood as the Reciprocal of the Ratio; which makes the Series either increasing or decreasing, according as r is greater or lesser than 1. And therefore tho' we call r the Ratio, yet if we understand it as the reciprocal Quote of the Antecedent divided by the Consequent, there will be no Ambiguity or Hazard of Error; and hereby we shall save the Trouble of making different Representations for increasing and decreasing Series. Yet in some Cases it will be convenient to represent them differently; and then taking the Ratio always as the Quote of the greater Term

Term divided by the leſſer, which muſt be a whole or mixt Number; and calling that Ratio r, an increaſing Series will be thus repreſented, $a : ar : ar^2$, &c. and a decreaſing thus, $a : \frac{a}{r} : \frac{a}{r^2} :$, &c. Wherefore if we repreſent a Series in the firſt manner, it's ſuppoſed to be taken indifferently for increaſing or decreaſing, unleſs it's expreſly ſaid to be increaſing; but the other manner does always expreſs a decreaſing Series.

From theſe Rules and Expreſſions of Geometrical Series we have the following

COROLLARIES.

1. If the leſſer Extreme of a Series is the Ratio, then the other Terms are the ſeveral Powers of the Ratio, thus: $r : r^2 : r^3 : r^4$, &c. or $A : A^2 : A^3 : A^4$, &c. For here r or A is the Ratio, and $r \times r = r^2$; $r^2 \times r = r^3$, and ſo on: Whence it's plain, that the Series of the Powers of any Number is a continued Geometrical Progreſſion.

2. If 1 is the firſt Term of a Series, the ſecond Term is the Ratio; and all the ſucceeding Terms are the ſeveral Powers of the Ratio. Thus, $1 : r : r^2$, &c. for the Ratio of 1 to r being r, the ſecond Term muſt be r^2, the third r^3, and ſo on.

3. Betwixt 1, and any Power of a Number as r^n, there fall as many Geometrical Means as the Index of that Power leſs 1, *viz.* $n - 1$, in the Ratio of 1 to the Root; for every Term after 1 is ſuch a Power of the ſecond, whoſe Index is equal to the Number of Terms after 1 to that Power; the thing is evident in the univerſal Series, $1 : r : r^2 : r^3 : r^4$, &c.

4 Every Geometrical Series, whoſe firſt Term is not 1, is equal to ſuch a Series multiplied by the given firſt Term; the Ratio of that Series being the ſame with that of the given Series. Thus: $A : Ar : Ar^2 : Ar^3 : Ar^4 :$, &c. is no other than the Products of this Series, $1 : r : r^2 : r^3 : r^4 :$, &c. multiplied by A.

5. As any Progreſſion may be thus repreſented, $A : Ar : Ar^2 : Ar^3 :$, &c. it's manifeſt that the Index of the Ratio in every Term expreſſes the Diſtance of that Term from the firſt A: And hence it follows immediately, that

6. Every Term is equal to the Product of the firſt, by that Power of the Ratio whoſe Index is the Diſtance of that Term from the firſt; and reverſly, the firſt Term is equal to the Quote of any greater Term divided by that Power of the Ratio, whoſe Index is the Diſtance of that Term from the firſt. So, for *Example*, the firſt Term being A, the Ratio r, and the Diſtance of any Term from A being d, that Term is Ar^d; and if you call that Term L, then is $L = Ar^d$; and reverſly, $A = L \div r^d$.

SCHOLIUM. Becauſe the greater Extreme of a Series is, by what's now ſhewn, equal to Ar^d, or Ar^{n-1}, (A being the leſſer Extreme, r the common Ratio, and d the Diſtance of the Extremes equal to $n - 1$, the Number of Terms leſs 1) and every Term below having the Ratio involved in it to one degree leſs than the preceding greater Term; therefore a decreaſing Series, which, when the greater Extreme is called l, is repreſented thus, $l : \frac{l}{r} : \frac{l}{r^2} : \frac{l}{r^3} :$, &c. may alſo be repreſented thus: $Ar^d : Ar^{d-1} : Ar^{d-2} :$, &c. Or thus, $Ar^{n-1} : Ar^{n-2} : Ar^{n-3} :$, &c. (becauſe $d = n - 1$) going on ſo till the Index of r be equal to 1, and then we have Ar the Term next to A.

7 But again more univerſally: From the ſame Expreſſion of a Series, it's manifeſt that the Difference of the Indexes of the Ratio in any two Terms expreſſes the Diſtance of theſe two Terms; for in every Step aſcending, the Ratio is involved once more than in the preceding; and therefore from any Term to any other, it's as much oftner involved in the greater than in the leſſer, as their Diſtance expreſſes; *that is*, the Difference of their Indexes is their Diſtance; and hence it follows immediately, that

8. Any Term of a Geometrical Series is equal to the Product or Quote of any other lesser or greater Term multiplied or divided by such a Power of the Ratio, whose Index is the Distance of these Terms; and any Term divided by any other lesser Quotes such a Power of the common Ratio, whose Index is the Distance of these Terms. For any Term being expressed Ar^d, (d being the Distance of this Term from A) any greater Term must have a Power of r, whose Index exceeds d by the Distance of these Terms (by the last) so, that Distance being m, the greater Term is Ar^{d+m}; but $r^d \times r^m = r^{d+m}$; (*Th.6. Ch.*1. *B.*3.) consequently, $Ar^{d+m} = Ar^d \times r^m$; and reversly, $Ar^{d+m} \div r^m = Ar^d$, and $Ar^{d+m} \div Ar^d = r^m$. Or this Truth may be deduced from *Coroll.* 6. Thus: Any Part of a Series, *i. e.* from any Term to another is still a Geometrical Series, whereof these two Terms are the Extremes; which being called A, L, and their Distance d, it's shewn that $A = L \div r^d$, and $L = Ar^d$; and lastly, $L \div A = r^d$.

Exam. In this Series, $2:6:18:54:162:486$. If we compare 6 and 162, whose Distance is 3; then is $6 = 162 \div$ Cube of 3, or 27; and reversly, $162 = 6 \times 27$; and lastly $162 \div 6 = 27$.

Schol. The immediate Use and Application of this last Truth we have in the Solution of these *Problems*.

(1.) Having any one Term of a Series and the Ratio, to find a Term at any Distance from the given one, without finding all the intermediate ones: The Solution of which is plainly contained in this *Coroll.* Thus: Take that Power of the Ratio, whose Index is the given Distance; and by it multiply or divide the given Term, and you have the Term sought above or below the given one. Thus: Any Term of a Series being multiply'd or divided by the fourth Power of the Ratio, gives a Term, which is the fourth above or below after the given one.

(2.) Having any two Terms of a Series and their Distance, to find any other Term at any Distance from either of the given ones; which is solved thus: Divide the greater by the lesser of the given Terms, the Quote is a Power of the Ratio, whose Index is the Distance of the given Terms. Suppose this Distance to be d, and extract the d Root of that Quote, (by the Methods explained, or referred to in *Book* III.) it is the Ratio sought: (all which is immediately contained in this *Coroll.*) Then having the Ratio, by it and any given Term we can find any other Term at any given Distance from the given Term, as in the preceding *Problem*.

Observe, For some Cases of this *Problem* there is a more easy Solution, which you'll find in *Theo.* XX. see the 3d Article of the *Schol.* after the 4th *Coroll.* as particularly, suppose the Term sought is lesser or greater then either of the Terms given, and its Distance from the lesser if it's greater than either of them, or from the greater if it's lesser, is a Multiple of the Distance of the given Numbers.

(3.) Of these three things, *viz.* the common Ratio of a Series, the Distance of any two Terms, and the Ratio of these two Terms; having any two we can find the third: for any Term being called A, r the common Ratio, and d the Distance of any greater Term from A, that Term is Ar^d, and the Ratio of A to Ar^d is r^d; which if we call R, then (1.) if d, r, are given, R ($=r^d$) is also known. (2.) If R and d are given, r is also known, *viz.* by extracting the d Root of $R = r^d$. (3. If r and R are given, d is also known, *viz.* by raising r to a Power which is equal to R ($=r^d$) and the Index of it is d.

9. The Sum of the Extremes of a Geometrical Series is equal to the Product of the lesser Extreme multiplied into the Sum of 1, and such a Power of the Ratio whose Index is the Distance of the Extremes. Thus, A being the lesser, L the greater Extreme, and d the Distance of the Extremes; $A + L = A + Ar^d$, for $L = Ar^d$; and consequently $A + L = A + Ar^d = A\,\overline{r^d + 1}$. Again; Because any two Terms of a Series may be considered as Extremes of a lesser Series, the same Rule will be good for expressing their Sum. Thus,

Thus, one Term being Ar^d, and another greater being Ar^{d+m}, their Sum is $Ar^d + Ar^{d+m} = Ar^d \times \overline{r^m + 1}$.

(4.) The Difference betwixt the Extremes of a Series is equal to the Product of the lesser multiplied by the Difference of 1 and such a Power of the Ratio whose Index is the Distance of the Extremes: Thus, $L - A = Ar^d - A = \overline{r^d - 1} \times A$. The same way may the Difference of any two Terms be expressed; so, $Ar^{d+m} - Ar^d = \overline{r^m - 1} \times Ar^d$; in all which the Indexes express the Distances of the Terms from the lesser Extreme A.

THEOREM XVIII.

THE Sums or Differences of every two adjacent Terms in a Geometrical Series make also a Geometrical Series, and in the same Ratio. Thus, if $A:B:C:D$, *&c.* is a Geometrical Series, so is this also $A+B:B+C:C+D$, *&c.* and this, $B-A:C-B:D-C$, *&c.* or $A-B:B-C:C-D$, *&c.*

Example: $2:4:8:16:32$, being a Geometrical Series, so are the Sums $6:12:24:48$, and the Differences $2:4:8:16$.

DEMONSTR. 1. For the Sums, It's true of any Series of four Terms, $A:B:C:D$. For $A:B::B:C$; therefore $A+B:B+C::A:B$. Again, $B:C::C:D$, hence $B+C:C+D::B:C$ (*Theor.* IV.) wherefore $A+B:B+C::B+C:C+D$; that is, $A+B:B+C:C+D$, are in the same Ratio of $A:B$, or $B:C$. By the same Reasoning, $B+C:C+D:D+E$ will be found to be in the continued Ratio of $B:C$, or $C:D$, so that $A+B:B+C:C+D:D+E$, are in continued Progression. And so the Reason proceeds thro' the Series *in infinitum.*

2. For the *Differences,* It's true of any Series of four Terms $A:B:C:D$; for since $A:B::B:C$, then (by *Theor.* V.) $B-A:C-B::A:B$, or $B:C$; and since $B:C::C:D$, then $C-B:D-C::B:C$. Hence $B-A:C-B:D-C$, are in continued Progression, and so on to more Terms.

Or we may shew the Truth of this *Theorem* yet more simply, thus; $A:Ar:Ar^2:Ar^3:Ar^4$, *&c.* being a Geometrical Progression; so is $A+Ar:Ar+Ar^2:Ar^2+Ar^3:Ar^3+Ar^4$; and $Ar-A:Ar^2-Ar$, *&c.* the continual Ratio of these being evidently r; for $\overline{A+Ar} \times r = Ar + Ar^2$, and so of the rest; also $\overline{Ar-A} \times r = Ar^2 - Ar$.

The *Reverse* of this *Theorem* is not true, for tho' the Sums of every two adjacent Terms of a Series are continuedly proportional, yet it will not always be true that that Series is so; for *Example:* $3:4:10:18$ are not $\div l$; yet $3+4:4+10:10+18$, or $7:14:28$, are so.

THEOREM XIX.

IF there be two Geometrical Progressions in the same Ratio, then any two Terms in the one, and any two equally distant in the other, are in the same Ratio, or Proportional.

Example: $2:4:8:16:32$; and $3:6:12:24:48$, are in the same Ratio; hence $4:16::3:12$, or $4:32::3:24$.

DEMONSTR. The Reason is plain from the last; for of Terms at equal Distance the greater is always the Product of the lesser by that Power of the Ratio whose Index is the Distance; therefore the Ratio and Distance being equal, these Terms are in the same Ratio to one another, which Ratio is that Power of the common Ratio of their respective Series. Universally, let one Series be $a:ar:ar^2:ar^3:ar^4$; and another $b:br:br^2:br^3:br^4$. Then $ar:ar^3::br^2:br^4$, the common Ratio here being r^2. Or thus, $ar^n:ar^{n+m}::br^o:br^{o+m}$, the Ratio being r^m.

COROLL.

Coroll. Hence having any two Terms of a Series, and any Term of another Series which has the ſame Ratio with the former, we can find a Term of this other as far diſtant from the given one as the two given Terms of the other Series are. The Application is plain.

Theorem XX.

Any two Terms of a Geometrical Progreſſion are :: *l* with any other two equally diſtant; alſo any three or more equally diſtant are ÷ *l*; ſo in this, 2 : 4 : 8 : 16 : 32 : 64 : 128, theſe are :: *l*, 2 : 4 :: 64 :: 128, and theſe, 2 : 16 :: 8 : 64: Again, theſe are ÷ *l*, 4 : 16 : 64, and theſe 2 : 8 : 32 : 128.

Demonstr. The *Reaſon* of this is the ſame as in the former *Theorem*, viz. the Equality of Diſtances which makes an Equality of Ratios betwixt the Numbers compared.

COROLLARIES.

1. If any two Terms are multiplied, and the Product divided by another Term, the Quote is equal to a fourth Term as far diſtant from one of the Terms multiplied on the one hand, as the Diviſor is from the other of them on the other hand. For ſince any four Terms are :: *l*, whereof the two leſſer are at the ſame Diſtance as the two greater, (and conſequently the leaſt and the next to the greateſt, at the ſame Diſtance as the greateſt and the next to the leaſt) therefore the Product of the two middle Terms of theſe four being divided by either of the Extremes, muſt quote the other; or the Product of the Extremes being divided by either of the Means, muſt quote the other; whence the *Corol.* is manifeſt.

Obſerve again, That if the Diviſor is one of the Extremes, *i. e.* lies on the ſame hand of (or is leſſer or greater than) either of the Terms multiplied, the Quote will be the other Extreme; *i.e.* will lie on the oppoſite hand of (or be contrarily greater or leſſer than) either of the Terms multiplied; conſequently the Diſtance of the Quote from the Diviſor will be the Sum of the Diſtances of both the Terms multiplied from the Diviſor (or from the Quote itſelf, which is at the ſame Diſtance): But if the Diviſor is one of the Means, *i. e.* lies betwixt (or is leſs than the one and greater than the other of) the Terms multiplied; and conſequently the Diſtance of the Quote from the Diviſor will be the Difference of the Diſtances of the Terms multiplied from the Diviſor, or from the Quote itſelf, which is the ſame Diſtance.

Example: In this Series, $a : b : c : d : e : f : g : h$, it's true that $a : c :: f : h$, where $cf \div a = h$, which is as far from f on the one hand, as a is from c on the other; and as far from a as the Sum of the Diſtances of c and f from a. Again; $ah \div c = f$, as far from h on the one hand, as c is from a on the other; alſo f is as far from c, as the Difference of the Diſtances of a and h from c.

Again; Becauſe it's the ſame thing in effect to multiply two Numbers, and divide their Product by another; or firſt to divide one of theſe two Numbers by the ſame Diviſor, and multiply the Quote by that other Number, for $\frac{bc}{a} = \frac{b}{a} \times c$, or $\frac{c}{a} \times b$. Therefore the laſt *Corollary* will be in effect the ſame if it's expreſſed in this manner, *viz.* if any Term of a Series is divided by another, and the Quote multiplied by a third Term, the Product is a Term as far diſtant from the Dividend or Multiplier on the one hand, as the other of theſe is from the Diviſor on the other hand; ſo that the Diſtance of the Product from the Diviſor will be the Sum of the Diſtances of the Dividend and Multiplier from the Diviſor, when the Diviſor is leſſer or greater than either of theſe; but if it lie between them, then the Diſtance of the Product from it will be the Difference of the Diſtances of the Dividend and Multiplier from the Diviſor; for $\frac{bc}{a} = \frac{b}{a} \times c$, which is now called a Product, is the very ſame Number that was before called a Quote; and what is here a

Dividend

Dividend (*viz.* b or c) is the ſame as was before one of the Multipliers, (and is in effect a Multiplier here alſo) and the Diviſor is the ſame as before.

2. If any Term of a Series is multiplied by itſelf, and the Product or Square divided by another Term, the Quote will be a Term of the Series as far diſtant from the Term ſquared on the one hand, as the Diviſor is from the ſame Term on the other hand; and conſequently the Quote will be as far from the Diviſor as twice the Diſtance of the Term ſquared from the Diviſor. Hence, *reverſly*, The Term which is in the middle betwixt two Terms is the Square Root of their Product. So in this Series, $a:b:c:d:e:f:g:h$, if $d \times d$ is divided by b, the Quote is equal to f, as far diſtant from d as d is from b, and twice as far from b as d is. The *Reaſon* is becauſe $b:d:f$ are $\div l$, and therefore $dd = bf$; whence, $dd \div b = f$, alſo $\overline{bf}|^{\frac{1}{2}} = d$.

3. Again, more univerſally; If any Term of a Series is raiſed to any Power, and that Power divided by ſuch a Power of any other Term whoſe Index is 1 leſs than that of the Dividend, the Quote is a Term of the Series whoſe Diſtance from the Root of the Diviſor is equal to the Product of the Index of the Power in the Dividend by the Diſtance of its Root from that of the Diviſor. The *Reaſon* will be plain from this *Example*: Let any two Terms of a Series be called $a:b$: a Series continued from theſe is (by *Probl.* III.) $a:b:\frac{b^2}{a}:\frac{b^3}{a^2}:\frac{b^4}{a^3}:$ *&c.* But, by the two preceding *Corollaries*, each of theſe Terms is a Term of any Series to which a, b can belong, ſince $\frac{b^2}{a}$ is a 3d $\div l$ to $a:b$, and each of the reſt a 4th to $a:b$, and the preceding Term. And here it's evident that each Term is twice or thrice, *&c.* as far from a as b is, according to the Index of b.

4. If any three or more Terms of a Geometrical Series are continually multiplied together, and the Product divided by ſuch a Power of a Term leſs than any of them, or greater, whoſe Index is 1 leſs than the Number of Terms multiplied, the Quote will be that Term of the Series whoſe Diſtance from the Root of the Diviſor is equal to the Sum of the Diſtances of theſe multiplied Terms from the ſame Root. The *Reaſon* of this is eaſily deduced from the *Theorem*, thus: Suppoſe any Term of a Series is a; and b, c, any two other Terms both greater or leſſer than a; then $\frac{bc}{a}$ expreſſes a Term diſtant from a by the Sum of the Diſtances of b and c (by this *Theorem*). Take another Term d, and multiply into the laſt found, it is $\frac{bc}{a} \times d = \frac{bcd}{a}$, which divided by a is $\frac{bcd}{a^2}$, which is a Number formed according to the Propoſition, and is alſo by this *Theorem* a Term of the Series as far diſtant from a as the Sum of the Diſtances of d and $\frac{bc}{a}$; but the Diſtance of $\frac{bc}{a}$ is the Sum of the Diſtances of b and c, therefore the Diſtance of $\frac{bcd}{a^2}$ is the Sum of the Diſtances of b, c, d; and it's obvious that the ſame Reaſoning will be good in the next Caſe, *i. e.* where four Terms are multiplied, and ſo *in infinitum*.

Or the *Demonſtration* of this Truth may be deduced without the *Theorem*, thus: Any Term of a Series may be called a, and if the common Ratio of the Series is r, then the ſeveral Terms after a will be ar, ar^2, ar^3, *&c.* ſo that any Term after a may be repreſented ar^n or ar^m, or with any other Index which will expreſs the Diſtance of that Term from a. Now if we take any two or more of theſe Terms after a, and multiply them together, the Product will be equal to the Product of theſe two Factors, *viz.* ſuch a Power of a whoſe Index is the Number of Terms multiplied; and ſuch a Power of r whoſe Index is the Sum of the Indexes of r in the ſeveral Terms multiplied (thus $ar^n \times ar^m = a^2 \times r^{n+m}$, and ſo of more Terms). Let us then ſuppoſe any Number of Terms after a multiplied

tiplied together, if that Number of Terms is n, and the Sum of the Indexes of r in the ſeveral Terms is m, then is the Product $a^n \times r^m$, which divide by a^{n-1}, the Quote is $a \times r^m$, a Term as far diſtant from a as m expreſſes, or as far as the Sum of the Diſtances of all the Terms multiplied.

Scholium. The immediate Uſe and Application of theſe *Corollaries* is in the Solution of the following *Problems*.

(1.) To find any Term of a Series by means of its Diſtance from the firſt Term, together with the firſt Term, and any two other, the Sum or Difference of whoſe Diſtance from the firſt is equal to the Diſtance of the Term ſought: The Solution of which is plainly contained in *Coroll.* 1. which need not to be repeated.

(2.) By the firſt and any three or more others, the Sum of whoſe Diſtances from the firſt is equal to the Diſtance of the Term ſought from the firſt, the Solution of which is in *Coroll.* 4.

(3.) By the firſt Term, and any one other whoſe Diſtance from it is an aliquot Part of the Diſtance of the Term ſought from the firſt; the Solution of which is contained in *Coroll.* 2. and 3.

(4.) When the Term ſought is in the very middle betwixt the given Terms, as in *Coroll.* 2.

Obſerve, If the Term ſought is betwixt the Terms given, but not in the middle, you have a Rule for ſolving this in *Probl.* 3. See Article 2. of the *Schol.* to *Coroll.* 8.

Theorem XXI.

In any Geometrical Series the Product of the Extremes is equal to the Product of any two middle Terms equally diſtant from the reſpective Extremes, (*i. e.* the leſſer Mean from the leſſer Extreme, and the greater from the greater) and to the Square of the middle Term, when the Number of Terms is odd. That is, theſe Products are all equal, *viz.* that of the Extremes, and theſe of every two Means, taken at equal Diſtance from the Extremes, and that of the middle Term by itſelf, *i. e.* its Square, when the Number of Terms is odd.

Example. 3 : 12 : 48 : 192. 168 : 3072 : 12288, being a Geometrical Progreſſion, theſe Products are equal, *viz.* $3 \times 12288 = 12 \times 3072 = 48 \times 168 = 192 \times 192$; each of them being $= 36864$.

Demonstr. This follows eaſily from the preceding, compared with *Theor.* I. for the *Extremes* of a Series, and any two Terms equally diſtant from them, are :: l by the preceding, and by *Theor.* I. the Products of the Extremes and Means of four Proportionals are equal; and the Product of the Extremes of three Numbers in continued Proportion is equal to the Square of the middle Terms; hence the thing to be proved is manifeſt.

Example. $a : b : c : d : e : f : g$, being a continued Series, theſe are proportional, $a : b :: f : g$; hence $ag = bf$. Again; $a : c :: e : g$; hence $ag = ce$. Alſo $a : d : g$ are $\div l$; hence $ag = dd$; wherefore $ag = bf = ce = dd$. However long the Series be, the Truth of the *Theorem* is equally clear.

Which we may alſo ſhew by this other Repreſentation: let a be the leaſt Extreme, r the Ratio, and d the Diſtance of the Extremes, then the greater Extreme is ar^d, (*Cor.* 6. *Prob.* 3.) and the Series of increaſing Terms from a, and of the decreaſing from ar^d, are as in the Margin. From whence it's obvious, that the Products of

$$\begin{array}{l}
a : ar : ar^2 : ar^3 : ar^4 : \&c.\ ar^d. \\
ar^d : ar^{d-1} : ar^{d-2} : ar^{d-3} : ar^{d-4} : \&c.\ a. \\
\hline
a^2 \times r^d
\end{array}$$

the ſeveral correſponding Terms ſtanding againſt one another in the two Series are all equal

to $a^2 \times r^d$; for r being involved once more in every Term from a, and once less in every Term from ar^d, makes still the same Product $a^2 \times r^d$.

Or it will be as clear if we represent a Series in this manner; $a:ar:ar^2$: &c. $:\frac{l}{r^2}:\frac{l}{r}$ $:l$, increasing from a for the one half of the Series, and decreasing from l for the other half: and if there is a middle Term let both Parts of the Series include it: And then the Truth proposed is obvious; each Product being $=al$. For tho' one of the Factors is a multiplied by some Power of r; yet the other is l divided by the same Power of r, which Multiplier and Divisor destroy one another in the Product proposed.

SCHOLIUMS.

1. Where a Series has an even Number of Terms, there are two Terms which we may call the two middle Terms, and in this Case the *Theorem* may be expressed thus: The Product of the two middle Terms, and of every two equally distant from them are equal: And we may also see this Truth in a Representation different from any of the former, thus; If $m:n$ be the two middle Terms, the Series ascending from n will be the continual Products of n by the Ratio, and descending from m it will be the continual Quotes of m divided by the Ratio, which in this Case is taken for the Quote of the greater Term divided by the lesser, as $n \div m$: Thus; *&c.* $:\frac{m}{r^3}:\frac{m}{r^2}:\frac{m}{r}:m:n:nr:nr^2:nr^3$: *&c.* In which the *Theorem* is manifest; for n being multiplied by any Power of r, and m being divided by the same, the Product of that Quote and Product is mn, the Divisor and Multiplier destroying one another: So $\frac{m}{r} \times nr = \frac{mnr}{r} = mn$; and universally, $\frac{m}{r^a} \times nr^a = \frac{mnr^a}{r^a} = mn$.

2. Where the Series has an odd Number of Terms, *i.e.* has a middle Term equally distant from both Extremes, then it's the same thing to say, The Products of the several Terms equally distant from the Extremes, or, The Products of Terms equally distant from the middle Term, are equal to another, and to the Square of the middle Term; and such a Series may be thus represented; *&c.* $\frac{m}{r^3}:\frac{m}{r^2}:\frac{m}{r}:m:mr:mr^2:mr^3$, *&c.* which also does clearly shew the Truth of the *Theorem*.

3. *Observe*, For the same Reasons here explained, the Product of any two Terms in a Series is equal to the Product of any other equally distant from the former two; and whereof one is taken as far above one of these former as the other is below the other of them: Because such four Terms are proportional; and of the Terms multiplied, the one Couplet are the Extremes, and the other the Means.

Also, The Product of any two Terms is equal to the Square of that Term which is in the midst, equally distant from either of them, because these three Terms are $::l$; so in the preceding Series $af=be$, because $a:b::e:f$; and $ae=cc$, because $a:c::c:e$.

Again; When a Series has an even Number of Terms, tho' the two middle Terms are not in the continued Ratio of all the rest above and below, yet the Products of the Extremes and middle Terms equally distant from them, will still be equal, because these Factors are $::l$, at least disjunctly.

CoROLL. The continual Product of any three or more Numbers in Geometrical Progression is a Power of some Order; particularly,

(1.) If the Number of Terms, n, is even (*i.e.* a Multiple of 2.) and the Extremes a, l, then suppose $n \div 2 = d$; I say, the continual Product of all the Terms of the Series is the d Power of the Product of the Extremes, or $\overline{al}|^d$: For since, as has been shewn,

 the

the Product of the Extremes, and of every two mean Terms equally distant from the Extremes, are all equal; and there are as many of these equal Products as the half Number of Terms, or d: Therefore the continual Product of all these equal Products (which is manifestly the continual Product of all the Terms in the Series) is equal to such a Power of any one of them (as the Product of the Extremes) whose Index is d. *Example*: Of this Series, $2:4:8:16:32:64$, the continual Product is $2\times4\times8\times16\times32\times64 = 2097152 = \overline{2\times64}|^3 = \overline{128}|^3$.

(2.) If the Number of Terms is odd, *i. e.* If there is a middle Term equally distant from each Extreme; suppose that middle Term is m, and the Number of Terms n; I say, the continual Product of all the Terms is the n Power of the middle Term, m, or m^n: For the Products of every two Terms equally distant from the middle Term, being equal to m^2 or $m\times m$; therefore, as to the continual Product of the whole Terms, the Series is the same in effect as if it were a Series of Terms all equal to m, in which case it's evident that the Product of the whole is m^n. *Example*: Of this Series, $2:4:8:16:32$, the continual Product is $2\times4\times8\times16\times32 = 32768 = 8^5$.

Observe, The continual Product of the Series, *viz.* $\overline{al}|^d$, or m^n, may also be a Power of as many other Orders as are denominated by the integral Numbers, which divide the Index d or n, without a Remainder, as has been already explained in *Book* III. Again; In some Cases the Product may be a Power of other Orders than what are assigned either by this *Corollary* or last *Observation*, according to the Nature and Composition of the 1st Term of the Series, and of the Ratio; but these not flowing immediately from the Nature of Progressions, are not to be considered here. This only I shall further *observe*, That if the Index n (*viz.* of the Power m^n the Product of an odd Series) is an odd Number, the Product cannot be a Power of the Order d. Supposing in this Case also $d = \frac{n}{2}$; because n being an odd Number, is not divisible by 2 without a Remainder; therefore $\frac{n}{2}$ or d is not the Index of a simple Power. Also, If n is an even Number the Product cannot be a Power of the Order n; for that Product being $\overline{al}|^d$, if this is also a Power of the Order n, then d is divisible by n without a Remainder, which is impossible, because $d = \frac{n}{2}$.

Theorem XXII.

In any Geometrical Progression, which soever of the Extremes you call the 1st Term, the Difference of the 1st and 2d Term is, to the Difference of the Extremes, in the same Ratio as the 1st Term is to the Sum of the whole Series, except the last Term: Or also, as the 2d Term to the Sum of the whole Series, except the 1st.

Example: In this Series, $2:6:18:54:162$, it is as 4 $(=6-2)$ to 160 $(=162-2)$ so is 2 to 80 $(=2+6+18+54)$; or take the Series decreasing, it is as 108 $(=162-54)$ to 160 so is 162 to 240 $(=162+54+18+6.)$ Again; it is $4:160::6:240$; and $108:160::80$.

Universally: If a Geometrical Series is $a:b:c:d:e$, &c. l, for the whole Sum put s, and let a be the 1st, and l the last Term; so that $s-a$, $s-l$, express the Sum of the whole Series, except the 1st or last Term; then is it $b-a$ (or $a-b$) $:l-a$ (or $a-l$) $::a:s-l::b:s-a$.

Demonstr. 1. Let us suppose the 1st Term to be the lesser Extreme, and the thing to be demonstrated is, that $b-a:l-a::a:s-l::b:s-a$. Thus; Of any Number of similar and equal Ratios, the Sum of all the Antecedents is to the Sum of all the Consequents as any one of the Antecedents to its Consequent (by *Theor.* IV, *Coroll.* 1.): But

But in case of a continued Progression all the Terms except the Last are Antecedents, and all except the 1st are Consequents; so that it is $s-l:s-a::a:b$; and *divisively* $\overline{s-a}-\overline{s-l}:b-a::s-l:a::s-a:b$; but $\overline{s-a}-\overline{s-l}=s-a-s+l=l-a$; therefore $l-a:b-a::s-l:a::s-a:b$; or, *reversly*, $b-a:l-a::a:s-l::B:s-a$.

2. If we suppose a to be the greater Extreme, the *Demonstration* proceeds the same way; only instead of $b-a$ and $l-a$, it is $a-b:a-l::a:s-l::b:s-a$.

COROLLARIES.

1. Having the Extremes of a Series, and the second Term, (or that next either of the Extremes) we can find the Sum of the whole Series, without knowing or finding any more of the Terms. Thus:

Multiply the Difference of the Extremes by the first Term, and divide the Product by the Difference of the first and second Term, the Quote is the Sum of all the Series except the last Term; to which Quote add the last Term, and the Sum is the thing sought. For since $b-a:l-a::a:s-l$, therefore $s-l=\frac{\overline{l-a}\times a}{b-a}$, and $s=\frac{\overline{l-a}\times a}{b-a}+l$. Or instead of multiplying by the first Term, multiply by the second Term; and you'll find the Sum of all the Series, except the first Term. For $b-a:l-a::b:s-a$; therefore $s-a=\frac{\overline{l-a}\times b}{b-a}$, and $s=\frac{\overline{l-a}\times b}{b-a}+a$. If we suppose b less than a, yet the Rule is the same, and the Reason of it also; by putting $a-b$ and $a-l$ in place of $b-a$ and $l-a$.

Again: If we take either of these Expressions for the Sum, *viz.* $\frac{\overline{l-a}\times a}{b-a}+l$, or $\frac{\overline{l-a}\times b}{b-a}+a$; and reduce them, by the common Rules, to a more simple Expression, we shall find it to be this, $s=\frac{bl-a^2}{b-a}$; *i. e.* multiply the last and second Terms, and from the Product subtract the Square of the first Term, and divide the Remainder by the Difference of the first and second Term, the Quote is the Sum. And the Truth of this we may also deduce otherwise. Thus, $a:b::s-l:s-a$, (as we saw above) and the Product of Extremes and Means are equal, *viz.* $as-a^2=bs-bl$. To each add bl, and it is $bl+as-a^2=bs$; and subtracting as, it is $bl-a^2=bs-as=\overline{b-a}\times s$; and dividing by $b-a$, it is $\frac{bl-a^2}{b-a}=s$.

2. From this Expression of the Sum, *i. e.* from this Equality, $as-a^2=bs-bl$; we have also the Solution of these *Problems*, *viz.* (1.) Having b, s, l to find a, which is, $a=\frac{s}{2}\pm\overline{\frac{s^2}{4}-\overline{bs-bl}}\Big|^{\frac{1}{2}}$ (*Prob.* 6. *C.* 2. *B.* III.) (2.) Having a, s, l to find b, which is, $b=\frac{as-a^2}{s-l}$.

(3.) Having a, s, b to find l, which is $l=\frac{bs-as+a^2}{b}$. For since $bl-a^2=bs-as$. Add a^2 to each, it is $bl=bs-as+a^2$; and dividing by b, it is $l=\frac{bs-as+a^2}{b}$. (4.) Having $a, b, s-l$ to find $s-a$, which is, $s-a=\frac{\overline{s-l}\times b}{a}$; which flows immediately from $as-a^2=bs-bl$. (5.) Having $a, b, s-a$ to find $s-l$, which is, $s-l=\frac{\overline{s-a}\times a}{b}$.

In the ſame manner having any three of theſe four, *viz.* a, $\overline{b-a}$, $\overline{l-a}$, $\overline{s-l}$, we can find the fourth; thus, $s-l=\frac{\overline{l-a}\times a}{b-a}$, (which is above demonſtrated, and from which follow the reſt, *viz.*) $b-a=\frac{\overline{l-a}\times a}{s-l}$; $l-a=\frac{\overline{s-l}\times\overline{b-a}}{a}$; and $a=\frac{\overline{s-l}\times\overline{b-a}}{l-a}$. Again: Having any three of theſe four, *viz.* b, $\overline{b-a}$, $\overline{l-a}$, $\overline{s-a}$; we can find the fourth thus, $s-a=\frac{\overline{l-a}\times b}{b-a}$. Which is above demonſtrated; and from which follows, that $b-a=\frac{\overline{l-a}\times b}{s-a}$; $l-a=\frac{\overline{s-a}\times\overline{b-a}}{b}$; and $b=\frac{\overline{s-a}\times\overline{b-a}}{l-a}$.

Scholium. In every Geometrical Progreſſion theſe five things are conſiderable, *viz.* the two Extremes, the Ratio, the Number of Terms, and the Sum: From which a Variety of *Problems* ariſes, whereof theſe are the chief and moſt uſeful, in which are given any three of theſe things to find the other two. But there is one Caſe, *viz.* having the Sum, Number of Terms, and either Extreme, to find the other and the Ratio; the Solution of which is too much above the Method I am limited to: however, I ſhall not entirely paſs it over, but bring the Solution ſo far as my Method permits, and muſt leave the reſt to ſuperior Methods.

The Uſe of the Symbols *employed in the following* Problems.

$a=$ the leſſer Extreme. $l=$ the greater Extreme. $s=$ Sum of the whole Series.	$r=$ the Ratio, which is to be taken as the Quote of the greater Term divided by the leſſer, and conſequently the reciprocal Ratio taken the other way.	$n=$ Numb. of Terms. $d=n-1=$ the Number of Terms leſs 1, or the Diſtance of the Extremes.

Problem IV.

Having the Extremes and Ratio A, l, r, *to find the Sum and Number of Terms* s, n.

SOLUTIONS.

1. For the Sum. It may be done various ways; thus,

1ſt *Method*: Having a, r, we have alſo b the ſecond Term, for $b=ar$; then by a, b, l we may find s, as in *Cor.* 1. *Theor.* 22. thus, $s=\frac{bl-a^2}{b-a}$.

But without finding b, we have various other ways.

2d *Method*: Divide the Difference of the Extremes by the Ratio leſs 1, the Quote is the Sum leſs the greater Extreme. Thus, $s-l=\frac{l-a}{r-1}$, and $s=\frac{l-a}{r-1}+l$.

Example: $a=3$, $l=48$, $r=2$; then is $s-l=45=\frac{48-3}{2-1}$, and $s=45+48$; as in this Series, $3:6:12:24:48$.

Demonstr. By *Theor.* 22. it is $b-a:l-a::a:s-l$, or $b-a:a::l-a:s-l$. But $b=ar$, therefore $b:a::r:1$; and diviſively, $b-a:a::r-1:1$. Hence $r-1:1::l-a:s-l$, and $s-l=\frac{l-a}{r-1}$; alſo $s=\frac{l-a}{r-1}+l$.

3d *Me-*

3d *Method.* Multiply the Difference of the Extremes by the Ratio, and divide the Product by the Ratio leſs 1, the Quote is the Sum wanting the leſſer Extreme. Thus; $s-a=\frac{\overline{l-a}\times r}{r-1}$; and hence $s=\frac{\overline{l-a}\times r}{r-1}+a$.

Example: $a=3$, $l=48$, $r=2$; then is $s-a=90=\frac{\overline{48-3}\times 2}{2-1}$, and $s=93$.

DEMON. Since $b:a::r:1$, (as before); therefore $b-a:b::r-1:r$. But $b-a:b::l-a:s-a$, (*Theorem* 22.) Hence $r-1:r::l-a:s-a$, and $s-a=\frac{\overline{l-a}\times r}{r-1}$, and $s=\frac{\overline{l-a}\times r}{r-1}+a$.

4th *Method.* Take the Product of the leſſer Extreme and the Ratio out of the greate Extreme, and divide the Difference by the Ratio leſs 1, the Quote is the Sum wanting both Extremes; thus, $s-a-l=\frac{l-ra}{r-1}$, and $s=\frac{l-ra}{r-1}+a+l$.

Exam. In the preceding Series, $s-a-l=42=\frac{48-6}{2-1}$, and $s=42+51=93$.

DEMONSTR. By the 2d *Method*, $s-l=\frac{l-a}{r-1}$; hence $s-l-a=\frac{l-a}{r-1}-a=\frac{l-a-ar+a}{r-1}=\frac{l-ar}{r-1}$. Or thus; If ar (which is the ſecond Term, when a is the firſt) be made the firſt Term; then, of ſuch a Series $s-l=\frac{l-ar}{r-1}$, by the ſecond *Method*: But this is plainly $=s-l-a$ of that Series which begins with a. Therefore, *&c.*

5th *Method.* Multiply the greater Extreme by the Ratio, and from the Product take the leſſer Extreme; divide the Difference by the Ratio leſs 1, the Quote is the Sum; thus, $s=\frac{rl-a}{r-1}$.

Exam. In the preceding Series, $s=\frac{2\times 48-3}{2-1}=93$.

DEMONSTR. This Rule is deduced from the 2d Method, thus, $s=\frac{l-a}{r-1}+l=\frac{l-a+rl-l}{r-1}=\frac{rl-a}{r-1}$; and it may be deduced the ſame way from the 3d or 4th Methods. Or we may deduce it from the 1ſt Method, by putting ar in the Place of b, thus, $\frac{arl-aa}{ar-a}=\frac{rl-a}{r-1}$, by dividing Numerator and Denominator by a. Or, again, from the 2d Method, Thus; If one Term more is joined to that Series whoſe greateſt Term is l, that new Term is rl; and putting rl in place of l, in *Rule* 2. it is $s-rl=\frac{lr-a}{r-1}$: But $s-rl$ in this new Series is equal to s in the former, whoſe greater Extreme was l, wherefore its Sum is $\frac{rl-a}{r-1}$.

But we may alſo demonſtrate this Rule independently of any of the preceding, thus: $a:b::s-l:s-a$ (as has been ſhewn in *Theor.* 22.), but $1:r::a:b$; hence $1:r::s-l:s-a$, and $s-a=rs-rl$, and $s-a+rl=rs$; alſo $rl-a=rs-s$; and, laſtly, $\frac{rl-a}{r-1}=s$.

$$\begin{array}{r} a : ar : ar^2 : \&c. : ar^{n-1} : ar^n \\ r - 1 \\ \hline ar + ar^2 + ar^3 : \&c + ar^n + ar^{n+1} \\ -a - ar - ar^2 - ar^3 : \&c. - ar^n. \\ \hline -a + ar^{n+1} \text{ or } ar^{n+1} - a \end{array}$$

Or also thus, (independently of *Theorem* 22.) Let the Series be expressed thus, $A : Ar : Ar^2 : \&c. : Ar^n$, and let it be multiplied by $r - 1$, it's manifest from the Work in the Margin, that the Product is $= Ar^{n+1} - A$; and if this Product is divided by $r - 1$, the Quote is $\frac{ar^{n+1} - a}{r - 1} = \frac{rl - a}{r - 1}$; because $ar^n = l$, and $rl = ar^{n+1}$.

2. For the Number of Terms n, Raise the Series from a till you find a Term equal to l, and then you'll have n. The *Reason* is obvious

Or thus: Divide the greater by the lesser Extreme, and raise the Ratio to a Power equal to that Quote, its Index is equal to $n - 1$, or d; because $l = ar^d$ (*Cor.* 6. *Probl.* III.) therefore $\frac{l}{a} = r^d$; so in the preceding Series, $48 = 3 \times 2^4 = 3 \times 16$; wherefore 5 is the Number sought.

Observe, There is another Method of finding n, *viz.* By the *Logarithms*: And tho' the Nature and Use of these Numbers is not yet explained, I shall here, nevertheless, deliver the Rule, that what belongs to Progressions may be found together; but the *Demonstration* and *Application* must both be referred till *Logarithms* are explained. See *Page* 507.

Rule. To find n by *Logarithms*, Subtract the *Log.* of a from that of l, and divide the Remainder by the *Log.* of r, the Quote is equal to d or $n - 1$; thus, $d = \frac{\text{Log.}\, l - \text{Log.}\, a}{\text{Log.}\, r}$. But *observe*, In this and all the following *Solutions* by *Logarithms*, that the Answer will not always be accurately true; and therefore to be sure of an exact Answer, the former Method is to be chosen.

Problem V.

Having the Extremes a, l, and Number of Terms n, to find the Sum s and Ratio r.

1. For the Ratio. Divide the greater Extreme by the lesser; extract that Root of the Quote whose Index is $n - 1$, or d, it is the Ratio. Thus, $r = \overline{l \div a}|^{\frac{1}{d}}$.

Example. $a = 3$, $l = 48$, $n = 5$, then is $r = \overline{48 \div 3}|^{\frac{1}{4}} = \overline{16}|^{\frac{1}{4}} = 2$.

Demonstr. $l = Ar^d$ (*Coroll.* 6. *Probl.* III.) therefore $l \div a = r^d$, and $r = \overline{l \div a}|^{\frac{1}{d}}$ (*Ax.* 1. *Book* III.)

Scholium. As to the Extraction of this Root, see what is said in *Book* III.; and if the Root is *Surd*, the *Problem* is impossible in rational Numbers.

2. For the Sum. Find it by a, l, r, as above, *viz.* $s = \frac{rl - a}{r - 1}$.

Or instead of r put its Equal $\overline{l \div a}|^{\frac{1}{d}}$; and so the Rule will be $s = \underline{\overline{l \div a}|^{\frac{1}{d}} \times l - a} \div \underline{\overline{l \div a}|^{\frac{1}{d}} - 1}$. Where *observe*, That the Expression for r standing here distinctly by itself, this is no other thing than bidding you first find r, and then by a, l, r, find s. But this Expression may be reduced to another Form wherein the Expression of r is lost, thus; $\underline{\overline{l \div a}|^{\frac{1}{d}} \times l - a} \div \underline{\overline{l \div a}|^{\frac{1}{d}} - 1} = \underline{l \times l^{\frac{1}{d}} - a \times a^{\frac{1}{d}}} \div \underline{l^{\frac{1}{d}} - a^{\frac{1}{d}}}$. The Manner and Reason of which Reduction

duction is this, $\overline{l \div a}|^{\frac{1}{d}} = l^{\frac{1}{d}} \div a^{\frac{1}{d}}$ (*Theor.* 4. *Book* III.), and therefore $\overline{l \div a}|^{\frac{1}{d}} \times l = l \times l^{\frac{1}{d}} \div a^{\frac{1}{d}}$; from which subtract a and it is $\underline{l \times l^{\frac{1}{d}} - a \times a^{\frac{1}{d}}} \div a^{\frac{1}{d}}$. Again; $\overline{l \div a}|^{\frac{1}{d}} - 1 = \underline{l^{\frac{1}{d}} \div a^{\frac{1}{d}}} - 1 = \underline{l^{\frac{1}{d}} - a^{\frac{1}{d}}} \div a^{\frac{1}{d}}$. And the former Remainder, divided by this, quotes $\underline{l \times l^{\frac{1}{d}} - a \times a^{\frac{1}{d}}} \div \underline{l^{\frac{1}{d}} - A^{\frac{1}{d}}}$. If you express all the Divisions here fraction-wise you will perceive the Method of the Reduction more easily. Yet here again *observe*, that tho' the *Problem* be possible in rational Numbers, it cannot be solved in this Form unless l and a are both Powers the Order d. And if it be here objected, That either this Solution is false, or the *Problem* impossible, since a true Solution must necessarily give the Answer to a possible Problem, this Difficulty is removed by considering that this new Expression is only the Effect of taking $l^{\frac{1}{d}} \div a^{\frac{1}{d}}$ for $\overline{l \div a}|^{\frac{1}{d}}$; which last may be rational (and must be so if the Problem is possible) tho' $l^{\frac{1}{d}}$ and $a^{\frac{1}{d}}$ are both Surd, (as has been shewn in *Book* III.) and the Consequence we are to draw from $l^{\frac{1}{d}}$ and $a^{\frac{1}{d}}$ being Surd, is not that the Problem is impossible in rational Numbers; but rather this, that making these Extractions further and further *in infinitum*, and applying them in this Rule, we shall approach infinitely nearer to the Answer of the Problem.

Problem VI.

Having the Extremes a, l, and Sum s, to find the Ratio r, and Number of Terms n.

1. For the Ratio. Divide the Difference of the Extremes by the Difference of the Sum and greater Extreme, the Quote is the Ratio less 1: Or thus; Divide the Difference of the Sum and lesser Extreme, by the Difference of the Sum and greater Extreme, the Quote is the Ratio: thus, $\frac{l-a}{s-l} = r - 1$, or $r = \frac{s-a}{s-l}$.

Exam. $a = 3$, $l = 48$, $s = 93$, then is $r - 1 = \frac{48-3}{93-48} = \frac{45}{45} = 1$, or $r = \frac{93-3}{93-48} = 2$.

Demonstr. $s - l = \frac{l-a}{r-1}$ (*Probl.* IV.), hence $r - 1 = \frac{l-a}{s-l}$ (the 1st *Rule*); hence $r = \frac{l-a}{s-l} + 1 = \frac{l-a+s-l}{s-l} = \frac{s-a}{s-l}$ (the 2d *Rule*). Which is also demonstrated thus; $s-l : s-a :: a : b :: 1 : r$; hence $r = \frac{s-a}{s-l}$.

2. For n. Find it by a, l, r, as in *Probl.* IV.

Or by *Logarithms*, thus: Divide the Difference of the Logarithms of l and a by the Difference of the Logarithms of $s-a$ and $s-l$, the Quote is $n-1$ or d: Thus $d = \frac{\text{Log. L} - \text{Log. } a}{\text{Log. } \overline{s-a} - \text{Log. } \overline{s-l}}$.

Coroll. Having r and the Difference of the Sum from either of the Extremes, we can find its Difference from the other Extreme; for since $r = \frac{s-a}{s-l}$, therefore $s - l = \frac{s-a}{r}$, and $s - a = \overline{s-l} \times r$.

PROBLEM VII.

Having either of the Extremes a or l, with the Ratio r, and Number of Terms n, to find the other Extreme l or a, and the Sum s.

1. If a is given, multiply it by the d Power of r, the Product is l: thus, $l = ar^d$. If l is given, then divide it by r^d, the Quote is a: Thus $a = l \div r^d$.
Example: $a = 2$, $r = 3$, $n = 5$; then is $l = 2 \times 3^4 = 2 \times 81 = 162$. And l being given, then is $a = 162 \div 81 = 2$.
DEMONSTR. By *Coroll.* 6. *Probl.* III. it is $l = ar^d$, and hence $a = l \div r^d$.

2. For the Sum. Apply a, l, r, as in *Probl.* IV. thus, $s = \frac{rl - a}{r - 1}$. Or it may be found otherwise in the given Terms, thus:

(1.) If a is given, then is $s = \frac{ar^n - a}{r - 1}$. For $l = ar^{n-1}$, therefore $rl = ar^{n-1} \times r = ar^n$, whence $s = \frac{rl - a}{r - 1} = \frac{ar^n - a}{r - 1}$.

(2.) If l is given, then is $s = \frac{lr^n - l}{r^n - r^{n-1}}$. For $a = \frac{l}{r^{n-1}}$, and $rl - a = rl - \frac{l}{r^{n-1}} = \frac{r^{n-1} \times rl - l}{r^{n-1}} = \frac{lr^n - l}{r^{n-1}}$; therefore $s = \frac{lr^n - l}{r^{n-1}} \div \overline{r - 1} = \frac{lr^n - l}{r^n - r^{n-1}}$.

PROBLEM VIII.

Having the Sum s, Number of Terms n, and Ratio r, to find the Extremes a, l.

1. For a, Raise r to the n Power; multiply the Sum by $r - 1$, and divide the Product by $r^n - 1$, the Quote is a: Thus, $a = \frac{s \times \overline{r - 1}}{r^n - 1}$.

Example: $r = 3$, $n = 5$, $s = 242$; then is $a = \frac{242 \times 2}{3^5 - 1} = \frac{484}{242} = 2$; as in this Series, $2 : 6 : 18 : 54 : 162$.

DEMONSTR. $s = \frac{rl - a}{r - 1}$ (*Probl.* IV.), and $l = ar^{n-1}$ (*Probl.* III. *Coroll.* 6.); hence $rl = r \times ar^{n-1} = ar^n$, and $rl - a = ar^n - a = a \times \overline{r^n - 1}$; wherefore $s = \frac{a \times \overline{r^n - 1}}{r - 1}$, and $s \times \overline{r - 1} = a \times \overline{r^n - 1}$, and lastly $a = \frac{s \times \overline{r - 1}}{r^n - 1}$.

2. For the greater Extreme, Find it by a, r, n, as in *Probl.* VII.
Or also in the given Terms, thus, find the n and $n - 1$ Powers of r, and multiply their Difference by the Sum; then divide the Product by $r^n - 1$, the Quote is the greater Extreme, thus, $l = \frac{s \times \overline{r^n - r^{n-1}}}{r^n - 1}$.

Example: In the last Series, $r^n = 243$, and $r^{n-1} = 81$, and $l = \frac{242 \times \overline{243 - 81}}{242} = 243 - 81 = 162$.

2 DEMON.

DEMONSTR. This proceeds from putting $\frac{s \times \overline{r-1}}{r^n - 1}$ in the place of a in the other Rule, *viz.* wherein $l = ar^{n-1}$, for thus, $l = \frac{s \times \overline{r-1}}{r^n - 1} \times r^{n-1} = \frac{s \times \overline{r-1} \times r^{n-1}}{r^n - 1}$; but $r \times r^{n-1} = r^n$; hence $\overline{r-1} \times r^{n-1} = r^n - r^{n-1}$; wherefore, lastly, $l = \frac{s \times r^n - r^{n-1}}{r^n - 1}$.

PROBLEM IX.

Having either Extreme, a, or l, the Sum s, and Ratio r, to find the other Extreme l, or a, and Number of Terms n.

1. For the unknown Extreme. And

(1.) If l is given, to find a. Multiply the Difference of the Sum and greater Extreme, by the Ratin less 1; subtract the Product from the greater Extreme, the Remainder is the lesser Extreme. Thus; $a = l - \overline{s-l} \times \overline{r-1}$; which being reduced gives also this other Rule, *viz.* $a = rl + s - rs$.

Example: $s = 242$, $l = 162$, $r = 3$, then is $A = 162 - 80 \times 2 = 162 - 160 = 2$.

DEMONSTR. In *Probl.* IV. *Method* 2. it is shewn that $s - l = \frac{l-a}{r-1}$, hence $\overline{s-l} \times \overline{r-1} = l - a$, and $a = l - \overline{s-l} \times \overline{r-1} = rl + s - rs$.

(2.) If a is given, to find l. Multiply the Difference of the Sum and lesser Extreme by the Ratio less 1, and divide the Product by the Ratio, then to the Quote add a, the Sum is l; thus $l = \frac{\overline{s-a} \times \overline{r-1}}{r} + a$; which being reduced is also $= \frac{rs + a - s}{r}$.

Example: $s = 242$. $a = 2$, $r = 3$, then is $l = \frac{240 \times 2}{3} + 2 = \frac{480}{3} + 2 = 160 + 2 = 162$.

DEMONSTR. In *Probl.* IV. it's shewn that $s - a = \frac{\overline{l-a} \times r}{r-1}$ hence $\overline{s-a} \times \overline{r-1} = \overline{l-a} \times r$, and $l - a = \frac{\overline{s-a} \times \overline{r-1}}{r}$, and $l = \frac{\overline{s-a} \times \overline{r-1}}{r} + a = \frac{rs + a - s}{r}$.

2. For the Number of Terms, Apply a, l, r, by *Probl.* 4.

Or by Logarithms, thus: (1.) If a is given, then subtract the Logarithm of r from that of $rs + a - s$; and from this Remainder subtract again the Logarithm of a: This last Remainder divide by the Logarithms of r, the Quote is $n - 1$: thus, $n - 1 = \frac{\text{Log.}\ \overline{rs + a - s} - \text{Log.}\ r - \text{Log.}\ a}{\text{Log.}\ r}$.

(2.) If l is given, From the Logarithm of l take the Logarithm of $rl + s - rs$, and divide the Remainder by the Logarithm of r, you have $n - 1 = \frac{\text{Log.}\ l - \text{Log.}\ \overline{rl + s - rs}}{\text{Log.}\ r}$.

PROBLEM X.

Having either Extreme, a, or l, the Sum s, and Number of Terms n, to find other Extreme, l or a, and the Ratio r.

1. For the unknown Extreme. Take the Difference of the Sum and given Extreme; raiſe it to the $n-1$ Power, which multiply by the given Extreme: This Product is equal to the Product of the $n-1$ Power of the Difference of the Sum and unknown Extreme, multiplied by the unknown Extreme; thus, $l\times\overline{s-l}^{n-1}=a\times\overline{s-a}^{n-1}$; wherefore, to finiſh the Solution, we muſt find a Number which being taken from s, and the Remainder raiſed to the $n-1$ Power, and this Power multiplied by that ſame Number, the Product ſhall be equal to $a\times\overline{s-a}^{n-1}$ if l is ſought; or equal to $l\times\overline{s-l}^{n-1}$ if a is ſought. But how to diſcover ſuch a Number is the greateſt Difficulty: Nor does my Method and Limits allow me to give any farther Direction about it. Therefore I ſhall only illuſtrate the Rule by an *Example*, and demonſtrate the Solution ſo far as it is deduced.

Example: $s=242$, $n=5$, $a=2$, then is $s-a=240$, $\overline{s-a}^{n-1}=240^4=3317760000$, which multiplied by $a=2$, is $6635520000=a\times\overline{s-a}^{n-1}$, and the Number ſought is $162=l$; for $s-l=242-162=80$, and $\overline{s-l}^{n-1}=80^4=40960000$, and $l\times\overline{s-l}^{n-1}=162\times40960000=6635520000$.

Demonstr. $l=ar^{n-1}$ (*Coroll.* 6. *Probl.* 3.), and $r=\frac{s-a}{s-l}$ (*Probl.* 6.) Hence $l=a\times\overline{s-a\div s-l}^{n-1}=a\times\overline{s-a}^{n-1}\div\overline{s-l}^{n-1}$; therefore $l\times\overline{s-l}^{n-1}=a\times\overline{s-a}^{n-1}$.

2. For the Ratio. (1.) If a is given. Find a Number, whoſe n Power ſubtracted from its Product by $\frac{s}{a}$ ſhall leave a Number equal to $\frac{s-a}{a}$, that Number is the Ratio; *i.e.* $\frac{sr}{a}-r^n=\frac{s-a}{a}$.

Demon. $l=ar^{n-1}$, (*Cor.* 6. *Prob.* 3.) and $l=\frac{rs+a-s}{r}$ (*Prob.* 9.); therefore $ar^{n-1}=\frac{rs+a-s}{r}$. Hence $ar^n=rs+a-s$, and $ar^n+s-a=rs$; alſo $s-a=rs-ar^n$; and laſtly, $\frac{s-a}{a}=\frac{rs-ar^n}{a}=\frac{rs}{a}-r^n$.

(2.) If l is given, find r ſo that $\frac{sr^{n-1}}{s-l}-r^n=\frac{l}{s-l}$.

Demon. By *Prob.* 7. $s=\frac{lr^n-l}{r^n-r^{n-1}}$; hence $sr^n-sr^{n-1}=lr^n-l$; and $sr^n-sr^{n-1}+l=lr^n$; alſo $l=lr^n-sr^n+sr^{n-1}=sr^{n-1}-sr^n+lr^n=sr^{n-1}-\overline{s-l}\times r^n$; and laſtly, $\frac{l}{s-l}=\frac{sr^{n-1}}{s-l}-r^n$.

TABLE

TABLE *of the last Seven* PROBLEMS, *with their Solutions.*

Prob.	Given.	Sought.	SOLUTIONS.
4	a, l, r	s, n	$s = \frac{l-a}{r-1} + l = \frac{rl-ra}{r-1} + a = \frac{l-ar}{r-1} + a + l = \frac{rl-a}{r-1}$. $n-1 =$ the Index of that Power of r which is $= \frac{l}{a}$, or $n-1 = \frac{\log. l - \log. a}{\log. r}$
5.	a, l, n	r, s	$r = \overline{l \div a}\vert^{\frac{1}{d}}$. s as above, or $s = \overline{l \div a}\vert^{\frac{1}{d}} \times l - a \div \overline{l \div a}\vert^{\frac{1}{d}} - 1$.
6.	a, l, s	r, n	$r = \frac{l-a}{s-l} + 1 = \frac{s-a}{s-l}$, and $n-1 = \frac{\log. l - \log. a}{\log. \overline{s-a} - \log. \overline{s-l}}$.
7.	a or r, n l	l or s a	$l = ar^{d}$ $a = l \div r^{d}$ s, as above; or $s = \frac{ar^{n} - a}{r-1}$. $s = \frac{lr^{n} - l}{r^{n} - r^{n-1}}$.
8.	s, n, r	a, l	$a = \frac{s \times \overline{r-1}}{r^{n} - 1}$. $l = \frac{s \times \overline{r^{n} - r^{n-1}}}{r^{n} - 1}$.
9.	a or s, r l	l or n a	$l = \frac{\overline{s-a} \times \overline{r-1}}{r} + a = \frac{rs + a - s}{r}$. $n-1 = \frac{\log. l - \log. \overline{rl + s - rs}}{\text{Log. } r}$ $a = l - \overline{s-l} \times \overline{r-1} = rl + s - rs$. $n-1 = \frac{\log \overline{rs + a - s} - \log. r - \log. a}{\log. r}$
10	a or n, s l	l or r a	For the unknown Extreme, $l \times \overline{s-l}^{\,n-1} = a \times \overline{s-a}^{\,n-1}$. For the Ratio, $\frac{sr}{a} - r^{n} = \frac{s-a}{a}$, and $\frac{sr^{n-1}}{s-l} - r^{n} = \frac{l}{s-l}$.

SCHOL. What has been before observed upon Arithmetical Progressions holds also in Geometrical, *viz.* that any three Numbers taken at random, will not make a possible *Problem* in every Case; and therefore we shall here also consider and explain the Invention of these Numbers, and how by means of any two given things the other three may be found. And in the first place observe, That as we have some general Rules in Arithmeticals for inventing three things to make a *Problem* possible, so we have here also. But again; in these we found also a Variety of particular Rules for the Invention of any one of the three things separately, without being obliged to find any of the other two, (though that Invention is owing to the Rules for finding these others); whereas here, the Solutions of the preceding *Problems* do not afford us Rules for this purpose in every Case; and therefore we must be content with the general Rules. And for this Reason the Solution of the *Problems*, wherein two things only are given, are comprehended in these general Rules; such of them at least as are solvable upon our Method. Now the first thing I shall do here, is to give you the general Limitations, which the several things belonging to these *Problems* must have with respect to one another; which are these:

a may be any Number less than s, and not greater than l.
l any Number not less than a, nor greater than s.
r any Number not less than 1.
n any Integer greater than 1.
s any Number not less than a or l.

The

The Reaſons of theſe Limitations, and that there are no other, will be manifeſt upon a ſmall Attention to the Nature of a Geometrical Progreſſion.

Now for the Invention of any three of theſe things to make a poſſible *Problem.*

(1.) If the things to be invented are $a, n, r,$ or $l, n, r,$ or $s, n, r,$ they may be taken at pleaſure within the general Limitation; becauſe from any Number a or l, we may raiſe a Series in any Ratio. Again; Chuſing any Ratio and Number of Terms, the Sum may be any Number whatever, ſince this has no Dependance on r or n.

(2.) If theſe are to be invented, *viz.* $a, l, r,$ or $a, l, n,$ or $a, n, s,$ or $l, n, s,$ or $r, l, s,$ or $a, r, s.$ Then take ſuch one of the preceding Caſes, in which are any two of the things to be now invented; and with the three things of that Caſe find the remaining thing to be invented (by *Prob.* 7, or 8.)

(3.) If a, l, s are to be invented, you muſt find all the five, by taking, firſt, any one of the three Caſes in *Art.* 1. (in any of which you have one of the things now required) and by theſe find the other two.

For the *Problems*; wherein two things only are given, they are all poſſible, except three, *viz.* when $a, s,$ or $l, s,$ or a, l are given. For any other two they are both found in one of the Caſes of *Art.* 1. *viz.* $a, n, r,$ or $l, n, r,$ or $s, n, r.$ And therefore, ſince the Numbers of any of theſe three Caſes may be taken at pleaſure within the general Limits, this ſhews how one of the three things now required (by means of the two things given) may be found; and then by *Prob.* 7, or 8. the other two are found. For *Example:* Suppoſe a, r given, then is n alſo given; for we may take any Integer greater than 1; and by a, n, r we can find l, s by *Prob.* 7. But if $a, l,$ or $a, s,$ or l, s are given, we cannot find the other three; becauſe we have no Method of finding any one of them conſiſtently with the two given things; ſince theſe two are not two of the things in any of theſe Caſes wherein three things may be taken at pleaſure within general Limits, as a, n, r; l, n, r; s, n, r; which are the Foundation of all theſe Rules.

CHAP. IV.

Of the Compoſition *of* Ratios *and* Proportions, *and things depending thereon.*

DEFIN. 1. THE Ratio of one Number to another is ſaid to be compounded of the Ratios of two or more other Numbers compared to as many; when the Antecedent of the firſt Ratio (called the Compound) is to its Conſequent, in the Ratio of the Product of the Antecedents to the Product of the Conſequents of all the other Ratios or Numbers compared. Thus: The Ratio of a to b is ſaid to be compounded of the Ratios of c to d, and of e to f, providing that theſe Numbers be $::l$, *viz.* $a : b :: ce : df$. Or thus: Take Ratios fraction-wiſe, placing the Antecedent as Numerator over the Conſequent. Then if one Ratio is a Fraction (or Quote) equal to the Product of ſeveral other Ratios multiplied as Fractions (or Quotes), that one is ſaid to be compounded of theſe others; whether it be in the ſame Terms with that Product, or only in equivalent Terms. *Example:* 8 : 15 is a Ratio compounded of theſe Ratios 2 : 3 and 4 : 5; becauſe $\frac{8}{15} = \frac{2}{3} \times \frac{4}{5}$. Alſo 1 : 2 is compounded of 2 : 3, and 3 : 4; for $\frac{1}{2} = \frac{2}{3}$

$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$. Now that this Definition, or Character of a Compound Ratio, is the same in effect as the preceding, is evident; because if $\frac{1}{2} = \frac{6}{12}$, which is the last Character; then $1:2::6:12$, which is the first Character; universally, if $\frac{a}{b} = \frac{ce}{df}$, then $a:b::ce:df$.

2. Any Number, Integer or Fraction, is also said to be compounded of others, or to be a Composite of these others, to whose Product it is equal: Thus if $a = cde$, then is a composite of c, d, and e, and the compounding Numbers are called the Factors of the Composition.

3. Two Numbers are said to be *like* or *similar Composites*, when having an equal Number of Factors, they are all in the same Ratio, comparing the lesser Factor of the one Composite to the lesser of the other, and so on in order to the greatest. Thus; ab, cd are *like Composites*, if $a:c::b:d$, (or $a:b::c:d$); also abc, def are *similar Composites*, if $a:d::b:e::c:f$.

Hence it is evident, that all similar Powers are similar Composites, the Roots being the compounding similar Factors. Thus; aa and bb are similar Composites, because $a:b::a:b$.

SCHOLIUM. The peculiar Doctrine of the *Composition* and *Resolution* of Numbers, you have in the following Book: But the last two Definitions were necessary here, because of some relative Properties of Numbers arising immediately from the Consideration of compound Ratios, and which are equally applicable to Integers and Fractions: whereas the Composition afterwards explained regards Integers only, because it's considered in Opposition to another thing which belongs not to Fractions, as they are distinguish'd from Integers.

4. Two Numbers, $a:b$, are said to be in the duplicate Ratio of other two $c:d$, when the former are as the Squares of the other; *i. e.* if $a:b::c^2:d^2$; and if $a:b$ are as the square Roots of $c:d$, *i. e.* if $a:b::c^{\frac{1}{2}}:d^{\frac{1}{2}}$; then $a:b$ are said to be in the subduplicate Ratio of cd. Again: If $a:b::c^3:d^3$, then $a:b$ are said to be in the triplicate Ratio of $c:d$; or if $a:b::c^{\frac{1}{3}}:d^{\frac{1}{3}}$, then are $a:b$ said to be in the subtriplicate Ratio of $c:d$. Universally, if $a:b::c^n:d^n$, or $c^{\frac{1}{n}}:d^{\frac{1}{n}}$; then $a:b$ is said to be in such a Ratio as is named from $c:d$, with a complex Denomination expressing such a Power or Root of $c:d$ as n expresses.

This was the antient way, and is still in Use; but it's plain, that it's as simple and convenient a way to name the Order of the Power or Root, and say, that $a:b$ are in the Ratio of the Squares or Cubes, or n Powers or Roots of $c:d$; which is yet easier expressed in Characters, thus, $a:b::c^n:d^n$, or $c^{\frac{1}{n}}:d^{\frac{1}{n}}$.

THEOREM I.

IN any Series of Numbers, whatever $::l$ or not, as A : B : C : D : E, &c. the first is to the last, in the Ratio compounded of the Ratios of all the intermediate Couplets, comparing always every Number as Antecedent to that which is immediately next it.

DEMONSTR. A is to E in the compound Ratio of all the intermediate Terms; that is, $\frac{A}{E} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} = \frac{ABCD}{EBCD}$; for BCD is common to both Numerator and Denominator, therefore both being divided by it, the Quotes A and E make an equal Fraction or Ratio $\frac{A}{E}$.

Now

Now however many Terms the Series has, the Product of the Numerators will be the Product of all the Series except the last, and the Product of the Denominators will be the Product of all the Series except the first; and therefore the Product of all the intermediate Terms will be an equal Fraction, and contain nothing in it but the first Term divided by the last.

THEOREM II.

ANY two Numbers whatever are to one another in a Ratio compounded of an indefinite Number of other Ratios, *i. e.* we can assign as many other Couplets as can be required, whose Ratios compounded shall be equal to the given Ratio.

DEMONSTR. *and* RULE. Consider the Difference betwixt the given Numbers, and multiply them, by such a Number that the Difference of the Products shall be at least equal to the Number of the Ratios to be found; then betwixt these Products you can take as many intermediate Numbers as will answer the Problem. For *Example:* To find 4 Ratios whose Compound is equal to $\frac{2}{31}$; multiply 2, 3, by 4, the Products are 8, 12; betwixt which there are these Numbers, 9, 10, 11, and the whole Series is 8, 9, 10, 11, 12. But $\frac{2}{3}$ is $= \frac{8}{12}$; and this, by the preceding *Theorem*, is $= \frac{8}{9} \times \frac{9}{10} \times \frac{10}{11} \times \frac{11}{12}$.

Now the general *Reason* of the Rule is this, The given Terms being equally multiplied, the Products are in the same Ratio, and their Difference contains the Difference of the given Numbers as oft as the Multiplier contains 1. Therefore 'tis plain, that betwixt the Products can be found as many intermediate Terms as their Difference hath Units less 1; so if you multiply by such a Number as makes the Difference of the Products as great as the Number of Ratios required, you can have as many intermediate Numbers as you need.

SCHOLIUM. If the Difference of the given Numbers is great enough for the purpose, there is no need to multiply, as if in the former *Example*, the Numbers given had been 8, 12, instead of 2, 3. But *observe*, That the greater you make the Difference (of Numbers in the same Ratio), the greater Variety of Choice you have in the intermediate Terms and Ratios: So if instead of 8, 12, I take 32, 48 (*i. e.* multiply by 16); then I can chuse any of these Series to solve the Question, *viz.* 32 : 33 : 35 : 36 : 48; or 32. 35. 36 . 40 . 48, *&c.*

COROLL. 'Tis plain, that one Ratio is not only equal to the Compound of an indefinite Number of others, but any Number required can be taken in an indefinite Variety.

PROBLEM I.

Any Number of Ratios (or Couplets of Numbers compared) being given, to continue them, *i. e.* to find a Series of Numbers, which shall be to one another (comparing them each to the next in order from the first to the last) in the given Ratios taken in any Order assigned.

RULE. Multiply the Terms of the first Ratio (*i.e.* that which ought to be betwixt the two first Terms of the Series) by the Antecedent of the second, *&c.* the Consequent of this by the Consequent of the first, and thus you have three Numbers which continue the two first Ratios. Then multiply this Series by the Antecedent of the third Ratio, and

 the

the Consequent of this by the last of that Series, and you have four Numbers which continue the first three Ratios. Go on thus, multiplying the last Series by the Antecedent of the next Ratio, and the Consequent of this by the last Term of that last Series. The Manner of the Work will be clear by an *Example:* Suppose these Ratios are to be continued in the Order proposed, *viz.* 2 : 3 . . 4 : 5 . . 6 : 7, they reduce to this Series 48 : 72 : 90 : 105. See the Work of *Example* 1.

The Method is the same if a Series is required in the continued Ratio of any two given Terms, so to continue the Ratio 2 : 3. See *Example* 2.

Exam. 1.	*Exam.* 2.
2 : 3	2 : 3
4 : 5	2 : 3
8 : 12 : 15	4 : 6 : 9
6 : 7	2 : 3
48 : 72 : 90 : 105	8 : 12 : 18 : 27

DEMONSTR. The *Reason* of this Rule will easily appear without many Words from the Operation; by which it's manifest that the Terms of each Ratio are equally multiplied, and consequently the Numbers produced continue in the same Ratio; for 2 : 3 are both multiplied by 4; then 4 : 5 are both multiplied by 3. Again; 8 : 12 : 15 are each multiplied by 6; and 6 : 7 both multiplied by 15; and so on.

SCHOLIUM 1. Here we have a new Method of raising a Series in a given Ratio, so as all the Terms be Integers: In which *observe*, That none of the Extremes can be given Numbers, for that is a Limitation which in many Cases will bring in Mixt Numbers into the Series, as we shall afterwards learn.

COROLLARIES.

1. Since the Series produced expresseth in Order the several given Ratios, The Extremes of it are in the compound Ratio of these given (by *Theor.* 1.) And for this Reason, in whatever different Order the same Ratios may be continued, tho' the Series produced will be different; yet the Extremes will not only be in the same Ratio, because they are the Compound of the given Ratios, but they will be the very same Numbers; because by the Rule, the first is always the Product of all the given Antecedents, and the last the Product of all the Consequents: So take the preceding Ratios in another Order, the Series is different, but its Extremes are the same, as in this Scheme,

4 : 5
6 : 7
24 : 30 : 35
2 : 3
48 : 60 : 70 : 105

2. 'Tis plain, that the Series raised will always have as many Terms as the Number of Ratios, and one more.

3. 'Tis evident, that the Extremes of a continued Series, (*i. e.* where all the given Ratios are the same Numbers) raised after this manner, are like Powers (of the Terms of the given Ratio) whose Index is the Number of Terms less 1, as in *Example* 2. where the Extremes 4, 9 are the Squares of 2 and 3; and 8, 27 are the Cubes or third Powers (3 being 1 less than 4, the Number of Terms); and in the same manner it will proceed *in infinitum*, because the lesser Extreme is multiplied by 2, and the greater by 3.

4. If the Extremes of the Series are only required, *i. e.* two Numbers which shall be in the compound Ratio of the Ratios given, then 'tis plain we have nothing more to do but multiply together all the Antecedents, and then all their Consequents, the two Products are in the Ratio sought; and if the same Ratio is continued, then raise the Antecedent and Consequent to a Power whose Index is the Number of Ratios (which is the Number of Terms that would be in the Series less 1.).

This follows also simply from the Definition of Compound Ratio.

SCHO-

SCHOLIUM. 2. This is to be remark'd, That tho' the Method of this *Problem* be applied to all the different Expressions or Variety of Terms in which any Ratio can appear, *i. e.* all the possible different Choices of two Terms in the same Ratio; yet we cannot hereby find all the Variety of different Series, which are the Continuation of the same Ratio; because some of the Series found this way being equi-multiplied or divided, will always make a new Series in the same Ratio, and which will often be different from any of the Series raised this way. For *Example*: 8 : 12 : 18 : 27 is a Series raised from 2 : 3; and if one is raised from 4 : 6, which is the same Ratio, it is 64 : 96 : 114 : 216; and if you go on, still taking higher Terms of the Ratio, the Series will consist of greater Numbers; yet betwixt these two Series raised from 2 : 3 and 4 : 6 we have several others: For by multiplying the first of them by these several Numbers, 2, 3, 4, 5, 6, 7, we have six new Series in the same Ratio different from these two, and consequently from all others that can be raised from any other Terms of the Ratio. And if it is required to find all the different Series that continue the same Ratio to a given Number of Terms, the *Rule* for this you'll find afterwards. (See *Schol.* 2. to *Probl.* 6. *Ch.* 1. *Book* V.)

PROBLEM II.

To reduce any Number of Ratios (given as in the last *Problem*) to a common Antecedent or Consequent, that is, to find a Series whose first Term compared to all the rest as a common Antecedent, or to whose last Term as a common Consequent all the rest being compared, their Ratios shall be equal to certain given Ratios, taken in a certain Order.

RULE. Take the Ratios fractionally, and reduce them (by the Rule of Fractions) to one common Denominator for a common Consequent, or to a common Numerator for a common Antecedent: Then place the Terms in a Series according to the Order assigned.

Exam. These Ratios, $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{7}{8}$, reduced to one Antecedent or Numerator are $\frac{56}{84}\cdot\frac{56}{70}\cdot\frac{56}{64}$. So the Series is 56 . 84 . 70 . 64; but reduced to one Consequent or Denominator they are $\frac{80}{120}\cdot\frac{96}{120}\cdot\frac{105}{120}$, and the Series is 80 . 96 . 105 . 120.

The Reason is manifest.

THEOREM III.

Part I. If one Rank of ::*ls* (whether continued or not) is multiplied by another Rank in the same or another Ratio, the Products are also ::*l*, in the compound Ratio of the Factors.

Example.
If A : B :: C : D
And $a : b :: c : d$
Then $Aa : Bb :: Cc : Dd$

DEMONSTR. AD = BC, and $ad = bc$, therefore $AD \times ad = Bc \times bc$. But $AD \times ad = Aa \times Dd$, and $BC \times bc = Bb + Cc$; hence $Aa \times Dd = Bb \times Cc$, wherefore $Aa . Bb :: Cc : Dd$: Or thus, $\frac{A}{B} = \frac{C}{D}$, and $\frac{a}{b} = \frac{c}{d}$ (by *Definitions*) therefore $\frac{A}{B} \times \frac{a}{b} = \frac{C}{D} \times \frac{c}{d}$ (*i. e.* equal Quantities multiplied by equal), that is $\frac{Aa}{Bb} = \frac{Cc}{Dd}$. Therefore $Aa : Bb :: Cc : Dd$. Or it may be shewn by another Representation, thus:

Example.
$A : Ar :: B : Br$
$C : Cn :: D : Dn$
$AC : ACrn :: BD : BDrn$

Wherein the common Ratio of the Products is plainly rn.

Exam. in Numbers.
3 : 4 :: 6 : 8
5 : 7 :: 15 : 21
15 : 28 :: 90 : 168

Part

Part II. If one Rank of $::ls$ is divided by another, The Quotes are $::l$ in a Ratio which is the Quote of the Ratio of the first Rank divided by the Ratio of the other. Thus in the preceding *Examples* it is $\frac{A}{a} : \frac{B}{b} :: \frac{C}{c} : \frac{D}{d}$.

DEMONSTR. $AD = BC$, and $ad = bc$, therefore $\frac{AD}{ad} \left(= \frac{A}{a} \times \frac{D}{d}\right) = \frac{BC}{bc}$ $\left(= \frac{B}{b} \times \frac{C}{c}\right)$; hence $\frac{A}{a} : \frac{B}{b} :: \frac{C}{c} : \frac{D}{d}$.

Or thus: Dividing each Antecedent by its Consequent (by the Rules of Fractions), the Ratio of the first Pair is $\frac{Ab}{Ba} = \frac{A}{B} \times \frac{b}{a}$, and the second is $\frac{Cd}{Dc} = \frac{C}{D} \times \frac{d}{c}$. But $\frac{A}{B} = \frac{C}{D}$, and $\frac{b}{a} = \frac{d}{c}$, by Supposition; therefore $\frac{A}{B} \times \frac{b}{a} = \frac{C}{D} \times \frac{d}{c}$; that is, the Ratios are equal and consequently the Terms are $::l$; also the Ratio is the Quote of the given Ratios, for $\frac{Ab}{Ba} = \frac{A}{B} \div \frac{a}{b}$, and $\frac{Cd}{Dc} = \frac{C}{D} \div \frac{c}{d}$.

But the whole will appear yet more easily in the other Representation, thus, $\frac{A}{C} : \frac{Ar}{Cn} :: \frac{B}{D} : \frac{Br}{Dn}$; for $\frac{Ar}{Cn} = \frac{A}{C} \times \frac{r}{n}$, and $\frac{Br}{Dn} = \frac{B}{D} \times \frac{r}{n}$; hence the Ratios are equal, and is also the Quote of the former Ratios, *viz.* r and n.

Exam. in Numbers.

2	: 5	::	6	: 15
7	: 9	::	14	: 18
14	: 45	::	84	: 270

For $14 \times 270 = 45 \times 84 = 3780$, also $\frac{14}{45} = \frac{2}{5} \times \frac{7}{9}$. If the Products are again divided by one of the former Series, it makes an *Example* of the second Part.

The *Reverse* of this *Theorem* is not universally true; for two Ranks may produce $::ls$, which are not themselves $::l$, as an Example will shew. This Rank, 6, 2, 2, 3, by this, 2, 9, 9, 9, produces this $::l$ Rank, $12 : 18 :: 18 : 27$.

SCHOLIUM. There are many Propositions demonstrable by the means of this *Theorem* alone, especially the 1st *Part*; which needing but a very small Help to demonstrate, I chuse to bring them here as *Corollaries*, and shall express them only in Characters, leaving you to express them in Words at length, and apply Numbers at pleasure, except the two last, which are of more frequent Use.

If $A : B :: C : D$, then follow these

COROLLARIES,	which arise from multiplying
1. $A^2 : B^2 :: AC : BD$	$A : B :: C : D$ by $A : B :: A : B$
2. $A^2 : C^2 :: AB : CD$	$A : C :: B : D$ by $A : C :: A : C$
3. $A^2 : BC :: AC : CD$	$A : B :: C : D$ by $A : C :: A : C$
4. $A^2 : AB :: CD : D^2$	$A : B :: C : D$ by $A : A :: D : D$
5. $A^2 : BC :: BC : D^2$	$A : B :: C : D$ by $A : C :: B : D$
6. $AB : AC :: BD : CD$	$A : C :: B : D$ by $B : A :: D : C$
7. $AB : BC :: BC : CD$	$A : B :: C : D$ by $B : C :: B : C$
8. $AC : BC :: BC : BD$	$A : B :: C : D$ by $C : C :: B : B$
9. $AB : AD :: AD : CD$	$B : A :: D : C$ by $A : D :: A : D$
10. $AC : AD :: AD : BD$	$C : D :: A : B$ by $A : A :: D : D$

COROLL. 11. $A^n : B^n :: C^n : D^n$; that is, if any four Numbers are $::l$, so are their Like Powers, whatever the Index be; the Deduction of which from the *Theorem* is plain; for if $A:B::C:D$ is multiplied by $A:B::C:D$, the Products are $::l$, *viz.* $A^2 : B^2 :: C^2 : D^2$; and if this again is multiplied by the same Rank, $A:B::C\,D$, and these Products also by the same, and so on, the last Product will still be a proportional Rank; and it's plain, they will be Like Powers of these Roots, $A:B::C:D$, the Index still increasing by 1 at every Multiplication.

The Truth contained in this *Corollary* may also be proved thus: Since $A:B::C:D$, then is $\frac{A}{B} = \frac{C}{D}$; therefore the Like Powers of these fractional Roots must be equal, which Powers are $\frac{A^n}{B^n} = \frac{C^n}{D^n}$; therefore $A^n : B^n :: C^n : D^n$: Or also $A\,D = B\,C$, and $\overline{AD}^n = \overline{BC}^n$: But $\overline{AD}^n = A^n \times D^n$, and $\overline{BC}^n = B^n \times C^n$; whence $A^n : B^n :: C^n : D^n$.

Hence again, The *Reverse* of this *Corollary* will easily be proved, *viz.* That if four Like Powers are $::l$, so are their Roots; for since $A^n : B^n :: C^n : D^n$, then $\frac{A^n}{B^n} = \frac{C^n}{D^n}$; and the n Roots of these Fractions are $\frac{A}{B} = \frac{C}{D}$, therefore $A:B::C:D$; or thus, $A^n D^n = B^n C^n$, but these are $\overline{AD}^n = \overline{BC}^n$ (by *Theor.* 1. *Ch.* 1. *Book* III.); and hence $A\,D = B\,C$, and $A:B::C:D$.

Again; It follows, That of a Series $\div l$, the Like Powers are also $\div l$, and *reversly.*

COROLL. 12. $A^2 : A\,D :: A\,D : D^2$; that is, the Product of two Numbers is a mean Proportional betwixt their Squares; for $A:D::A:D$, and $A:A::D:D$; which two Ranks multiplied produce $A^2 : A\,D :: A\,D : D^2$; or thus, $A^2 \times D^2 = A\,D \times A\,D$.

SCHOLIUM. This is a *Corollary* of the *Theorem*, but independent of the Supposition of $A:B::C:D$; and it has also a Demonstration independent of this *Theorem*; thus, $A:D :: A^2 : A\,D$, by Equi-multiplication. Again; $A:D::A\,D:D^2$; and hence, $A^2 : A\,D :: A\,D : D^2$.

Example: Take the Numbers 5 and 8, their Squares are 25 and 64, their Products 40, and $25:40::40:64$.

Or, All these *Corollaries* follow from the equal Products of Extremes and Means, founded all upon this, that $A\,D = B\,C$, and that $A\,D \times B\,C = \overline{AD}^2 = \overline{BC}^2$.

THEOREM IV.

TWO composite Numbers of an equal Number of Factors, are to one another in the compound Ratio of these Factors compared one to one in any Order.

DEMONSTR. Take any two Numbers, A, B, and suppose $A = abc$, and $B = mno$, then is $\frac{A}{B} = \frac{abc}{mno}$; But $\frac{abc}{mno} = \frac{a}{m} \times \frac{b}{n} \times \frac{c}{o}$, therefore $\frac{A}{B} = \frac{a}{m} \times \frac{b}{n} \times \frac{c}{o}$.

SCHOLIUM. This Proposition is the same in effect as this, *viz.* If the several Factors of the one are divided by those of the other, and the Quotes multiplied together, the Product is equal to the Quote of one whole Dividend by the other.

LEMMA.

When several Fractions (or Ratios) are continually multiplied together, the Product is equal to the Product of as many equivalent Fractions (or Ratios) expressed in any other different Terms.

Exam.

Example: If $\frac{2}{3}=\frac{4}{6}$, and $\frac{3}{5}=\frac{12}{20}$, then $\frac{2}{3}\times\frac{3}{5}=\frac{4}{6}\times\frac{12}{20}$, *viz.* $\frac{6}{15}=\frac{48}{120}$.

DEMONSTR. The *Reason* is manifest from this *Axiom,* That equal Fractions multiplied by equal, produce equal; *i. e.* equal Fractions of equal Fractions are equal.

COROLL. The continual Product of any Number of equal Fractions (or Ratios) expressed in different Terms, is equal to such a Power of any one of them as has for its Index the Number of Factors (or Fractions multiplied), that is, the Square if they are 2, the Cube, if 3, &c.

Exam. If $\frac{2}{3}=\frac{4}{6}$, then $\frac{2}{3}\times\frac{4}{6}=\frac{2}{3}\times\frac{2}{3}$, or the Square of $\frac{2}{3}$.

Again: If $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}$, then is $\frac{2}{3}\times\frac{4}{6}\times\frac{6}{9}=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$, or the Cube of $\frac{2}{3}$.

Universally: If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=$, &c. then is $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}\times\frac{g}{h}\times$, &c. $=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times$ &c. making as many Factors in this Part as the other; so that if the Number of Factors (or equivalent Fractions) is n, and any one of them is $\frac{a}{b}$, the Product of the whole is $=\frac{a^n}{b^n}$ (that is, $\frac{a}{b}$ raised to the n Power.)

THEOREM V.

TWO like composite Numbers are in the Ratio of the like Powers of any two of their similar Factors, the Index of the Power being equal to the Index of the Composition (or Number of Factors.) Thus:

If ab and AB are like Composites, *i. e.* $a:A::b:B$; then $ab:AB::a^2:A^2$: (i. e. ab to AB is in the duplicate Ratio of a to b); and if abc, ABC, are like Composites, they are as $a^3:A^3$ (or in the triplicate Ratio of a to A.) Generally; If $abcd$, &c. : ABCD, &c. are similar, let the Number of Factors be n; then it will be, $abcd$, &c. : ABCD, &c. $::a^n:A^n$.

DEMON. The Reason of this is plain from the last *Theorem,* and *Corollary* to the *Lemma.* For $\frac{abcd,\ \&c.}{ABCD,\ \&c.}=\frac{a}{A}\times\frac{b}{B}\times\frac{c}{C}\times\frac{d}{D}$, &c. (by *Theor.* 5.) and these Factors being, by Supposition, equal Fractions or Ratios, their Product is equal to $\frac{a^n}{A^n}$ by *Corollary* to the *Lemma*; that is, $abcd$, &c. : ABCD, &c. $::a^n:A^n$.

THEOREM VI.

IN every Progression, the first Term is to the last as such a Power of the first, whose Index is the Distance of the Extremes, or Number of Terms less 1, is to the like Power of the second Term; or as those like Powers of any two Terms of the Series next other. Thus: If the Number of Terms is 3 or 4, the Extremes are in the duplicate or triplicate Ratio (or as the Squares or Cubes) of the first and second Terms; and so of others. As in this Series, A : B : C : D : E : F : G, whose Number of Terms is 7, I say $A:G::A^6:B^6$. And *generally,* if we make A the first Term, B the second, and L the last of any such Series, and n the Number of Terms less one; then I say, $A:L::A^n:B^n$.

DEMON. By *Theorem* 2. A is to L in the Ratio composed of all the intermediate Ratios, *i. e.* $\frac{A}{L}=\frac{A}{B}\times\frac{B}{C}\times\frac{C}{D}$, &c. which are in Number equal to n, and all equal to one another; wherefore their Product or Compound is equal to the n Power of any one of

 them,

them, *viz.* of the first, or $\frac{A}{B}$ (by *Corol.* to *Lem.* preceding); that is, $\frac{A}{L}=\frac{A^n}{B^n}$; whence $A : L :: A^n : B^n$.

This is demonstrable also without the *Lemma*. Thus: If A is the first Term, the Ratio r, the Distance of the Extremes n; then is the second Term Ar, and the last Ar^n; and $A : Ar^n :: A^n : \overline{Ar}^n$ ($= A^n \times r^n$). For the Product of Extremes and Means are equal, *viz.* $A \times A^n \times r^n = A^n \times Ar^n$.

The *Reverse* is not true, *viz.* that if the first is to the last of a Series in this Ratio, the Series is $\div l$; for the changing any one of the Terms will shew this to be false.

Corol. If the Extremes and Number of Terms of a Geometrical Series are known, we learn from this *Theorem* a Rule, how by these to find the second Term. Thus: Multiply the last Term by such a Power of the first, whose Index is the Number of Terms less one; and divide the Product by the first Term, the Quote is the like Power of the second Term, whose Root being found is the thing sought. The Reason of this Rule is obvious; for since $A : L :: A^n : B^n$, therefore $A \times B^n = L \times A^n$, and $B^n = \frac{L \times A^n}{A}$ according to the Rule. But observe, Since $\frac{A^n}{A} = A^{n-1}$, therefore $B^n = L \times A^{n-1}$; which saves the Trouble of one Multiplication in raising A, and also the Division by A; and since n is here supposed to be the Number of Terms less one, therefore $n - 1$ is that Number less 2; *i. e.* it is the Number of Means betwixt A and L; and if we call $n - 1 = m$, then is $n = m + 1$. And to find the first of any Number of Means betwixt A and L, (or the Mean next that Extreme which is called the first, as A) which Mean call B; The Rule is thus: Multiply the last Term by the m Power of the first, and from the Product extract the $m + 1$ Root; which is thus, $B = \overline{A^m \times L}|^{\frac{1}{m+1}}$

Theorem VII.

If four Numbers are proportional, *viz.* $A : B :: C : D$; then as many Means as fall betwixt $A : B$ (in any Ratio as r) so many also fall betwixt $C : D$ in the same Ratio.

Demon. From A raise a Series in the Ratio r, as $A : Ar : Ar^2$, &c. and 'tis certain you must come at last to a Term equal to B; because 'tis supposed that B is the last Term of a Series raised from A in the Ratio r. Now n being the Number of Terms $- 1$, 'tis plain that the last Term of the Series is $ar^n = B$. Raise also a like Series from C; thus, $C : Cr : Cr^2 : Cr^3 :$ &c. C^n. These two Series being in the same Ratio, it will be $A : Ar^n :: C : Cr^n$; *that is*, $A : B :: C : Cr^n$: but $A : B :: C : D$; therefore $Cr^n = D$. Again; $C : Cr^n$ have as many Means as $A : Ar^n$ (*i. e.* as $A : B$) by the nature of the Series raised. Therefore, *lastly*, $C : D$ ($= C : Cr^n$) admit as many Means as $A : B$ ($A : Ar^n$).

The *Reverse* is also true; for if $A : B$ and $C : D$ admit an equal Number of Means in the same Ratio, they are in the same compound Ratio; *i. e.* $A : B :: C : D$.

Scholium. Here we do not distinguish betwixt Integers and Fractions, but take Numbers in general; yet when we speak of Means, we always understand rational Terms. In the next Book, you'll learn what Qualifications of the Extremes make the Means Integers or Fractions, and what do or do not admit of rational Means.

Theorem VIII.

Any two Numbers which are like Powers, as $A^n : B^n$:, admit as many Means $::ls$, as the Index less 1, (or $\overline{n-1}$,) in the Ratio of the Roots $A : B$; that is, 1 if they are square Numbers, 2 if they are Cubes, &c.

Demon. If a Series is raised (by *Problem* 1.) in the Ratio of $A : B$: to a Number of Terms equal to $n + 1$, the Extremes will be $A^n : B^n$: (by *Corol.* 2, and 3. to that *Problem*); and when the Number of Terms is $\overline{n+1}$, the Number of Means will be $\overline{n-1}$; therefore, &c.

The

The *Reverse* of this *Theorem* does not hold; as an Example will shew, *viz.* 16 : 24 : 36 : 54; wherein tho' there are two Means, yet the Extremes are not Cube Numbers. But if the lesser Extreme is 1, the Reverse must always hold; because the Terms of that Series are the several Powers of the second Term, (*Corol.* 2. *Prob* 3. *Chap.* 3.) Afterwards you'll find it demonstrated, that they are either such Powers, or Equi-multiples of such Powers.

SCHOLIUM. If it be required to fill up the Means betwixt A^n and B^n, knowing their Roots $A : B$; the whole Series may be raised by *Prob.* 1.

But I shall here add another Demonstration of this *Theorem*, from which we shall see more particularly the Composition of every Term in the Series, and a new Method of filling up the mean Terms required. Thus: Suppose a Series of the Powers of A^n : and B^n: beginning with 1; these are Geometrical Progressions in the Ratios of $1 : A$, and $1 : B$, and have an equal Number of Terms (by what has been already explained) Place these two Series the one above the other in a reverse Order, as in the following Example: Multiply the one Series into the other, *i. e.* every Term of the one into the correspondent Term of the other, and the Products make a Series of $n+1$ Terms, whose Extremes are $A^n : B^n$: in the continued Ratio of $A : B$, (*viz.* the Compound of the Ratios of the two Series multiplied) as has been already demonstrated in *Theor.* 3. and is indeed obvious in this first Case by Inspection into the Series itself. And here we see how every middle Term is composed of the Powers of the Roots; and so we have a new Rule for solving

$$\begin{array}{llllll} A^n : A^{n-1} & : A^{n-2} & : A^{n-3}, & \&c.\ A & : 1. \\ 1\ : B & : B^2 & : B^3, & \&c.\ B^{n-1} & : B^n. \\ \hline A^n : B A^{n-1} & : B^2 A^{n-2} & : B^3 A^{n-3}, & \&c.\ A B^{n-1} & : B^n. \end{array}$$

Problem 1. when the given Ratios are all equal; being, for Example, $A : B$ Or if the Extremes are given, and also the Roots, we can fill up the middle Terms more easily than by *Prob.* 1. by making up the Series of the Powers of A and B, and then multiply them together in this manner.

But the same Truth, as to the component Parts of the middle Terms, will also appear from the Method of *Prob.* 1. as here: Suppose the Ratio $A : B$ to be continued, 'tis done thus by *Prob.* 1. And 'tis plain that continuing it on, the Root A will be once more compounded in every Term at every Multiplication, and one Term more will be added, which will be the next higher Power of B.

$$\begin{array}{l} A : B \\ \quad A\ : B. \\ \hline A^2 : AB : B^2. \\ \qquad A\ : B. \\ \hline A^3 : A^2B : AB^2 : B^3,\ \&c. \end{array}$$

THEOREM IX.

ANY two like composite Numbers admit as many Means as the Index (or Number of component Factors) -1, and in the Ratio of their similar Factors. Thus:

If A and B represent two like Composites of two Factors, they admit 1 Mean; if of three Factors, they admit 2 Means, and so on: Generally, if they are composite of an n Number of Factors, they admit $n-1$ Means in the Ratio of any two of their similar Factors; suppose $a : b$. (*i. e.* a in the composite A, and b in B.)

DEMON. By *Theorem* 5. A and B are in the Ratio of the n Powers of any two of their similar Factors, *i. e.* $A : B :: a^n : b^n$. But (by *Theor.* 8.) $a^n : b^n$: admit $n-1$ Means in the constant Ratio of $a : b$, and (by *Theor.* 7.) $A : B$: admit as many in the same Ratio.

The Reverse of this *Theorem* is also true, but the Proof of it depends upon some things not yet explained; therefore it must be referred to another place. (See *Theor.* 34. in §. 11. *Chap.* 1. *Book* V.)

SCHOLIUM. If the component Factors are known, the mean Terms may easily be fill'd up, and their particular Compositions discover'd, after the Method of this Example. Suppose $abcde$, and $ABCDE$, the two like Composites, the Series will be this, $abcde$

: bc

$: bcde A : cde AB : de ABC : e ABCD : ABCDE$. In which the Means are filled up, by taking out the first Term of the lesser Extreme, and putting in the first of the greater; and so on in Order, taking out the next Term of the first Extreme, and putting in the next Term of the last Extreme, till all the Terms of the first Extreme are taken out; and that the Series thus made is $\div l$ in the Ratio of the similar Factors, is manifest by Inspection.

But if we only know the Ratio, (*i. e.* any one Couplet of the similar Factors) the Means must be filled up by the common Rule of *Prob.* 3. *Chap.* 3.

COROLLARIES.

1. The Product of two Numbers which are like Composites of two Factors, is a Square Number; so if N, M are similarly composed of two Factors, they admit one Mean, as X, and then 'tis plain, $NM = X^2$: Or, independently of this *Theorem*, it may be demonstrated thus: Let $N = ab$, and $M = AB$, and because they are similar, $a : b :: A : B$, and $aB = bA$; wherefore $aB \times bA$ ($= ab \times AB$) is a Square Number, whose Root is $aB = bA$.

The *Reverse* of this *Corollary* is also true, *viz.* That if two Numbers, A, B, produce a Square, they are Like Composites of two Factors; but the *Demonstration* of this belongs to another Place. (See *Coroll.* 1. *Theor.* 34. in §. 11. *Ch.* 1. *Book* V.)

2. Any two Like Composites of an even Number of Factors produce a Square Number; for any two of the Factors of the one, and the similar two of the other, produce a Square by the 1st *Coroll.* The like will any other two do; and these two Squares multiplied together will produce a Square (by *Theorem* 1. *Book* III.) and so on, taking in the two next Factors. *Example:* If there are four Factors, $abcd : ABCD$, because 'tis $a : b : A : B$, and $c : d :: C : D$; hence $abAB = \overline{aB}^2$, also $cdCD = \overline{cD}^2$: And, lastly, $abcd \times ABCD = \overline{aB}^2 \times \overline{cD}^2 = \overline{aBcD}^2$. Or thus; since $aB = bA$, and $cD = dC$, therefore $aBcD = bAdC$; and these multiplied produce $abcd\,ABCD = \overline{aBcD}^2$, or $\overline{bAaC}^2$.

3. If the Product of two Like composite Numbers of an odd Number of Factors is divided by the Product of any two of the similar Factors (one in each Composite) the Quotient is a Square Number. *Example:* If A, B, are two such Numbers, the least Number in A being a, and the least in B being b, then I say, $\frac{AB}{ab}$ is a Square Number; for when one Factor is taken out of each there remains an even Number of component Factors in each, whose Product makes a Square Number: But by dividing the total Product of all the Factors by the Product of any two similar ones, we effectually take them away, and reduce the Case to an even Number of Factors.

THEOREM X.

IF betwixt any Number A, and each of two others B and C, there falls an equal Number of Means, there fall as many betwixt B and C, which are in the Ratio of the 2d Terms of the two Series from A to B, and C.

DEMONSTR. Suppose the first Mean betwixt A and B is b, and the first betwixt A and C is c, then, by *Theorem* VI. these Proportions mark'd in the Margin are true (n being the Number of Terms $- 1$); and hence this is also true, $B : C :: b^n : c^n$; but (by *Theorem* 8.) betwixt $b : c^n$ there fall $\overline{n - 1}$ Means in the Ratio $b : c$, (*viz.* as many as fall by Supposition betwixt $A : B$ and $A : C$; for n being the Number of Terms $- 1$, the Number of Means is $n - 1$); therefore, (by *Theorem* 7.) there fall as many betwixt $B : C$, in the same Ratio of $b : c$.

$$\text{Series}\begin{cases} A : b \text{ - - - - - } B \\ A : c \text{ - - - - - } C \end{cases}$$

$$\text{hence}\begin{cases} A : B :: A^n : b^n \\ A : C :: A^n : c^n \end{cases}$$

SCHO-

SCHOLIUMS.

1. The Means that fall betwixt A : B and A : C must manifestly be in different Ratios; for they must be different Numbers, since there are an equal Number of them, and tend to different Extremes, B and C; but the Means betwixt B and C may be in the same Ratio with one of the former, or different from both; for if we suppose A : B :: B : C, then the Ratio in the Series from B to C will be the same as that from A to B.

2. The *Reverse* of this *Theorem* is this; If any Number of Means fall betwixt two Numbers, C, B, there is another Number, A, betwixt which, and each of these, C, B, there fall as many Means. But the *Demonstration* of this depends upon something not yet explained, and must be referred to another Place. (See *Theor.* 27. and 28. in §. 11. *Ch.* 1. *B.* V.)

THEOREM XI.

IF one Extreme of a Series is a Power whose Index n is the Number of Terms $-$ 1, the other Extreme is also a like Power. Or thus; If two Numbers, whereof one is an n Power, admit $\overline{n-1}$ Means, the other is also an n Power. If the one Extreme is A^n and the other B, I say, B has an n Root; *i. e.* If n is 2 then B is a Square Number. If $n = 3$ then B is a Cube Number, *&c.*

DEMON. Betwixt 1 and A^n there fall $\overline{n-1}$ Means, (*viz.* all the inferior Powers of A) and since there fall as many by Supposition betwixt A^n and B, therefore (by the last) there fall as many betwixt 1 and B; and hence, lastly, B is an n Power, *viz.* the n Power of the 2d Term of the Series; for all the Terms of a Series from 1 are the Powers of the 2d. (*Coroll.* 2. *Probl.* 3. *Ch.* 3.)

THEOREM XII.

OF four Numbers :: *l*, if any three of them are Like Powers, so is the 4th, and its Root is a 4th :: *l* to the Roots of the other three compared in the same Order as their Powers: Thus, if $a^n : b^n :: c^n : d$, I say, d is an n Power, and $a : b :: c : d^{\frac{1}{n}}$

DEMONSTR. Betwixt $a^n : b^n$, there fall $\overline{n-1}$ Means (*Theor.* 8.) and as many betwixt $c^n : d$ (*Theor.* 7.), therefore d is an n Power (*Theor.* 11.), which is the first Part. Again; Suppose $d = r^n$, so that $d^{\frac{1}{n}} = r$; then because $a^n : b^n :: c^n : r^n$, therefore $a : b :: c : r$ (*Theorem* 3. *Coroll.* 11.), that is, $a : b :: c : d^{\frac{1}{n}}$, because $r = d^{\frac{1}{n}}$, which is the last thing.

Or thus; Since $a^n : b^n :: c^n : d$, therefore $\frac{b^n c^n}{a^n} = d$; but $\frac{b^n c^n}{a^n} = \left.\frac{bc}{a}\right|^n$, therefore $\frac{bc}{a} = d^{\frac{1}{n}}$, or $a : b :: c : d^{\frac{1}{n}}$.

CHAP. V.

Containing the Comparison *of unequal Ratios (wherein the Ratio is always to be understood as the Quote of the Antecedent by the Consequent): With the Comparisons of* Arithmetical *and* Geometrical PROPORTIONS.

Observe, *For the Words* greater *and* lesser, *I use these Signs,* $\urcorner$ *and* $\llcorner$. *Thus,* A $\urcorner$ B, *or* A $\llcorner$ B, *signifies* A *greater or lesser than* B; *and* A : B $\urcorner$, *or* $\llcorner$ C : D, *signifies that the Ratio of* A *to* B *is greater or lesser than that of* C *to* D.

§. I. *The*

§. I. *The Comparison of* Unequal *Ratios*.

THEOREM I.

IF to each of two unequal Numbers, a, b, be added or subtracted an equal Number, the Ratio of the Sums is lesser, but the Ratio of the Differences greater, than that of the given unequal Numbers, comparing the greater Term to the lesser; or, contrarily, comparing the lesser to the greater: Thus, if $a \urcorner b$, then $a+c:b+c$, $\angle\, a:b$, and $a-c:b-c \urcorner a:b$. *Example:* $3 \urcorner 2$ and $3+4:2+4 \angle 3:2$, and $3-2:4-2 \urcorner 3:4$

DEMONSTR. If $a:b::c:d$, then $a+c:b+d::a:b$ (*Theor.* 4. *Ch* 3. *B.* IV.) but because $a \urcorner b$, therefore $c \urcorner d$ and $b+c \urcorner b+d$; wherefore $a+c:b+c \angle a+c:b+d$ (*Ax.* 11.), and consequently $a+c:b+c \angle a:b$.

But comparing the lesser to the greater, proceed thus; $b+d:a+c::b:a$; but $b+c \urcorner b+d$, therefore $b+c:a+c \urcorner b+d:a+c$; hence $b+c:a+c \urcorner b:a$.

Again; For the Differences, $a-c:b-d::a:b$, but $a \urcorner b$, hence $c \urcorner d$, and $b-c \angle b-d$, therefore $a-c:b-c \urcorner a-c:b-d$, and consequently, $a-c:b-c \urcorner a:b$: But comparing the lesser to the greater, then $b-d:a-c::b:a$; and since $d \angle c$, therefore $b-d \urcorner b-c$, and hence $b-c:a-c-b \angle b-d:a-c$, and consequently, $b-c:a-c \angle b:a$.

The *Reverse* is also true, *viz.* If $a+c:b+c \angle a:b$, therefore $a \urcorner b$; and so of the rest of the Parts, which is easily demonstrated.

THEOREM II.

IF four Numbers are $::l$, $a:b::c:d$, the Antecedents a, c, being less than their Consequents, and a, b, the two least of the four, then, by equally increasing or decreasing the two Antecedents, or the two Consequents, or both, the Proportion is destroyed, and the Ratio of the first Couplet so alter'd will be lesser or greater than that of the second in these Circumstances, *viz.*

1. By equal Addition to the Antecedents, the 1st Couplet has the greater Ratio; thus, $a+n:b \urcorner c+n:d$.

2. By equal Diminution of the Consequents, the 1st Couplet has the greater Ratio; thus, $a:b-n \urcorner c:d-n$.

3. By equal Addition to the Consequents, the Ratio of the 1st Couplet is least; thus, $a:b+n \angle c:d+n$.

4. By equal Diminution of the Antecedents, the Ratio of the 1st Couplet is least; thus, $a-n:b \angle c-n:d$.

5. By equal Addition to all the four, the Ratio of the 1st Couplet is the greatest; thus, $a+n:b+n \urcorner c+n:d+n$.

6. By equal Diminution of all the four, the Ratio of the first Couplet is greatest; thus, $a-n:b-n \urcorner c-n:d-n$.

7. By equally increasing the Antecedents and diminishing the Consequents, the Ratio of the 1st Couplet is the greater; thus, $a+n:b-n \urcorner c+n:d-n$.

8. By equally diminishing the Antecedents and increasing the Consequents, the Ratio of the 1st Couplet is the least; thus, $a-n:b+n \angle c-n:d+n$.

DEMONSTRATION.

Case 1. $a+n:b \urcorner c+n:d$. For $a+n:c+n \urcorner a:c$ (*Theor.* 1.) but $a:c::b:d$, therefore $a+n:c+n \urcorner b:d$; and hence (by *Theor.* 4th, below; which is demonstrated independently of this) $a+n:b \urcorner c+n:d$.

Case

Case 2. $a:b-n \urcorner c:d-n$. For $b-n:d-n \angle b:d$ (*Theor.* 1.); but $b:d::a:c$; hence $b-n:d-n \angle a:c$; or $a:c \urcorner b-n:d-n$; therefore $a:b-n \urcorner c:d-n$ (*Theor.* 7.)

Case 3. $a:b+n \angle c:d+n$. For $b:d::a:c$, and $b+n:d+n \urcorner b:d$ (*Theor.* 1.), consequently $b+n:d+n \urcorner a:c$, and (by *Theor.* 7.) $b+n:a \urcorner d+n:c$; and, *reversly*, $a:b+n \angle c:d+n$ (*Theor.* 6.)

Case 4. $a-n:b \angle c-n:d$. For $a:c::b:d$; but $a-n:c-n \angle a:c$, or $b:d$, i. e. $b:d \urcorner a-n:c-n$, and (by *Theor.* 7.) $b:a-n \urcorner d:c-n$; and, *reversly*, $a-n:b \angle c-n:d$.

Case 5. $a+n:b+n \urcorner c+n:d+n$. For multiplying the Antecedent of the one into the Consequent of the other, the Products which (are the Antecedents when the two are reduced to a common Consequent) are $ad+an+dn+nn$, and $bc+bn+cn+nn$; the 1st of which is the greater, because taking equal Parts out of both, *viz.* $ad=bc$, and $nn=nn$, there remains in the first $an+dn=n\times\overline{a+d}$, and in the second $bn+cn=n\times\overline{b+c}$; but $\overline{a+d} \urcorner b+c$ (*Theor.* 14.), therefore the 1st is greatest, *i. e.* $a+n:b+n \urcorner c+n:d+n$.

Case 6. $a-n:b-n \urcorner c-n:d-n$. For multiplying the Antecedent of each into the Consequent of the other, the Products are $ad-an-dn+nn$, and $bc-bn-cn+nn$; out of each take nn, and compare the Remainders, which are $ad-\overline{an+dn}$, and $bc-\overline{bn+cn}$; but $ad=bc$, and $a+d \urcorner b+c$, therefore the 1st is greatest, *i. e.* $a-n:b-n \urcorner c-n:d-n$.

Case 7. $a+n:b-n \urcorner c+n:d-n$. For $a+n:c+n \urcorner a:c$, or $b:d$ (*Theor.* 1.), and $b-n:d-n \angle b:d$ (*Theor.* 1.), therefore $a+n:c+n \urcorner b-n:d-n$, and $a+n:b-n \urcorner c+n:d-n$ (*Theor.* 7.).

Case 8. $a-n:b+n \angle c-n:d+n$. For $a-n:c-n \angle a:c$, or $b:d$ (*Theor.* 1.), and $b+n:d+n \urcorner b:d$; hence $a-n:c-n \angle b+n:d+n$; and lastly, $a-n:b+n \angle c-n:d+n$ (*Theor.* 7.).

Coroll. All these *Cases* are applicable to Numbers in Geometrical Progression. I shall only mention the Application of *Case* 5. and 6. thus: By Addition or Subtraction of the same Numbers, to or from each Term of a Series $\div l$, the Sums or Differences are not $\div l$; but (comparing them from the least to the greatest) does continually decrease; thus, if $a:b:c:d$, *&c.* are $\div l$, then $a+n:b+n \urcorner b+n:c+n \urcorner c+n:d+n$, *&c.* But from the greatest to the least, it does continually increase.

Theorem III.

If $a:b \urcorner c:d$, and $a \angle c$, or $a=c$, then is $b \angle d$.

Example: $6:7 \urcorner 8:13$, or $6:7 \urcorner 7:9$; also $6 \angle 8$, $7=7$, then $7 \angle 13$ or 9.

Demonstr. It cann't be that $b=d$; for since $a \angle$ or $=c$, then $a:b$ would be $\angle$ or $=c:d$, contrary to Supposition; and it would be yet more so if we suppose $b \urcorner d$.

The *Reverse* is also true, as you'll easily prove, *viz.* If $b \angle d$, and $a \angle$ or $=c$, then $a:b \urcorner c:d$.

Theorem IV.

If $a:b \urcorner c:d$, also $a+b=c+d$, then is $a \urcorner c$.

Example: $6:2 \urcorner 5:3$, and $6+2=5+3$, then $6 \urcorner 5$.

Demonstr. Suppose $a=c$, then must $b \angle d$ to make $a:b \urcorner c:d$; but if $a=c$, and $b \angle d$, then $a+b \angle c+d$, contrary to Supposition. And if we suppose $a \angle c$, it will yet more strongly follow that $a+b \angle c+d$.

The *Reverse* is also true, *viz.* If $a+b=c+d$, and $a \urcorner c$, then is $a:b \urcorner c:d$. The *Reason* is easy from the preceding Method.

THEOREM V.

IF $a:b \urcorner c:d$, then is $ad \urcorner bc$.

DEMONSTR. $c+n:d \urcorner c:d$; let n be taken so as $a:b::c+n:d$, and then $ad = bc+bn$, hence $ad \urcorner bc$.

Or this Truth is in effect already demonstrated in Fractions, where it's shewn, that if $\frac{a}{b} \urcorner \frac{c}{d}$, then $ad \urcorner bc$.

The *Converse* of this is also true, *viz.* If $ad \urcorner bc$, then $a:b \urcorner c:d$, for $\frac{ad}{b} \urcorner c$, and $\frac{a}{b} \urcorner \frac{c}{d}$, by equal Division.

THEOREM VI.

IF $a:b \urcorner c:d$, then *reversly*, $d:c \urcorner b:a$.

DEMONSTR. $\frac{a}{b} : \frac{c}{d} :: \frac{d}{c} : \frac{b}{a}$ (*Ch.* 1. *gen. Cor.* 13.) and if $\frac{a}{b} \urcorner \frac{c}{d}$, therefore $\frac{d}{c} \urcorner \frac{b}{a}$, which Consequence is also proved already from the Nature of Fractions.

Or thus, Take $a:b::c+n:d$, whence $d:c+n::b:a$; but $d:c \urcorner d:c+n$, therefore $d:c \urcorner b:a$.

THEOREM VII.

IF $a:b \angle c:d$, then, *alternately*, $a:c \urcorner b:d$.

DEMONSTR. $\frac{a}{b} : \frac{c}{d} :: \frac{a}{c} : \frac{b}{d}$ (*Ch.* 1. *gen. Cor.* 12.) and if $\frac{a}{b} \urcorner \frac{c}{d}$, therefore $\frac{a}{c} \urcorner \frac{b}{d}$, which is also already proved in the Doctrine of Fractions.

Or thus, Take $a::b:c+n:d$, whence $a:c+n::b:d$; but $a:c \urcorner a:c+n$, consequently $a:c \urcorner b:d$.

THEOREM VIII.

If $a:b \urcorner c:d$, then, *compoundly*, $a+c:b+d \angle a:b$, but $\urcorner c:d$; also $a+b:c+d \angle a:c$, and $\urcorner b:d$.

DEMONSTR. Take $a:b::c+n:d$, then $a+\overline{c+n}:b+d::a:b$; but $a+c:b+d \angle a+\overline{c+n}:b+d$, therefore $a+c:b+d \angle a:b$. Again; Take $c:d::a:b+n$, then $c+a:d+\overline{b+n}::c:d$; but $a+c:b+d \urcorner a+c:d+\overline{b+n}$, therefore $a+c:b+d \urcorner c:d$.

For the second Part: Since $a:b \urcorner c:d$, therefore, *alternately*, $a:c \urcorner b:d$ (*Theor.* 7.) and then by the 1st Part it is $a+b:c+d \angle a:c$, and $\urcorner b:d$.

The *Alternations* of all these Conclusions is true by vertue of the preceding *Theorem*.

The *Converse* of this *Theorem* is also true, *viz.* if $a+c:b+d \urcorner c:d$, then $a:b \urcorner c:d$; for, (by the following *Theorem*) $a+c-c$ $(=a):b+d-d$ $(=b)$ $\urcorner c:d$ (taking $a+c$ and $b+d$ instead of a, b, in the following *Theorem*), that is, $a:b \urcorner c:d$. The *Converse* of the other Conclusions will be found the same way.

THEOREM IX.

IF $a:b \urcorner c:d$, then, *divisively*, $a-c:b-d \urcorner a:b$, and also $\urcorner c:d$. Again, $a-b:c-d \urcorner a:c$, and also $\urcorner b:d$.

DEMONSTR. Take $a:b::c+n:d$, then $a-c-n:b-d::a:b$; but $a-c:b-d \urcorner a-c-n:b-d$, therefore $a-c:b-d \urcorner a:b$, and consequently also $\urcorner c:d$, which is $\angle a:b$. The other Part, or $a-b:c-d \urcorner a:c$, or $b:d$, is proved the same way, by taking $a:c \urcorner b:d$.

The

The *Couverse* is alſo true, and depends upon the 7th in the ſame manner as the Converſe of that upon this.

SCHOLIUM. As we have argued from the greater Ratio to the leſſer in the preceding five *Theorems*, ſo we may argue the ſame way from the leſſer to the greater; for this is all contained in the other; becauſe if $a:b \urcorner c:d$, then $c:d \angle a:b$, and it's no matter which of the two Ratios we ſuppoſe to be greateſt.

THEOREM X.

IF there are two Ranks of four Numbers each, whereof the Antecedents of the one are the Conſequents of the other Rank, as a, b, c, d, and b, e, d, f; and if $a:b \urcorner c:d$, alſo $b:e \urcorner d:f$, then is, $a:e \urcorner c:f$.

If $a:b \urcorner c:d$
and $b:e \urcorner d:f$
then $a:e \urcorner c:f$

DEMONSTR. $a:c \urcorner b:d$, and $b:d \urcorner e:f$ (*Theor.* 7.) hence $a:c \urcorner e:f$, and $a:e, \urcorner c:f$ (*Theor.* 7.)

THEOREM XI.

OF two Ranks of four Numbers each, where the Antecedents or Conſequents of both are the ſame Numbers, as in the Margin, and the Ratio of the firſt two greater than that of the other two, the Ratio of the Sums of the Antecedents to that of the Conſequents, is greater than the Ratio of the two common Terms.

If $a:b \urcorner c:d$
and $e:b \urcorner f:d$
then $\overline{a+e}:b \urcorner \overline{c+f}:d$
or
$a+e:c+f \urcorner b:d$

DEMONSTR. Take $a-n:b::c:d$, and $e-m:b::f:d$, then $\overline{a-n}+\overline{e-m}:c+f::b:d$ (*Theor.* 8. *Chap.* 3.), but $\overline{a+e} \urcorner \overline{a-n}+\overline{e-m}$, hence $a+e:c+f \urcorner \overline{a-n}+\overline{e-m}:c+f$, or $b:d$.

THEOREM XII.

OF two Ranks, whereof the Extremes of the one are the Means of the other, as in the Margin; if $a:b \urcorner c:d$, and $b:e \urcorner f:c$, then $a:e \urcorner f:d$.

$a:b \urcorner c:d$
$b:e \urcorner f:c$
$a:e \urcorner f:d$

DEMONSTR. $ad \urcorner bc$, and $bc \urcorner ef$ (*Theor.* 5.) hence $ad \urcorner ef$, and $a:e \urcorner f:d$, by Converſe of the 5th.

SCHOLIUM. If the Ranks are ſo diſpoſed that the Extremes or Means of the one are alſo the Extremes and Means of the other, as in the Margin; yet we can draw no Concluſion, becauſe all that immediately follows, is only that $ad \urcorner bc$, and $ef \urcorner bc$; but leaves it undetermined, whether ad is $=ef$, or not. for it may be either way; as theſe *Examples* ſhew, *viz.* (1.) $3:6 \urcorner 4:15$, and $9:6 \urcorner 4:5$, and $3 \times 15 = 5 \times 9$. (2.) $3:6 \urcorner 4:15$, and $8:6 \urcorner 4:5$, and $3 \times 15 \urcorner 8 \times 5$.

$a:b \urcorner c:d$
$e:b \urcorner c:f$

§. II. Arithmetical *and* Geometrical *Proportions compared.*

LEMMA.

IF four Numbers are proportional, either Arithmetically or Geometrically, the *leaſt* and *greateſt* of the four are the two Means or the two Extremes.

a, b, c, d

DEMONSTR. The leaſt of the four is either one of the Extremes or one of the Means. Suppoſe

 1. That

1. That a is the least, then because $a \angle b$, so is $c \angle d$; but c is also greater than a (which is the least of the four), and hence $d \urcorner b$; consequently d is the greatest.

2. If b is the least, then, taking them *reversly*, b, a, d, c, because the one Extreme is the least, the other is the greatest (by the 1st *Case*), *i. e.* in the other Position, the two Means are the least and greatest.

Coroll. Of four Numbers, $:l$, or $::l$, the two greatest or the two least, are one of them an Extreme, the other a Mean.

Theorem XIII.

If four Numbers are $::l$, the Difference of that Couplet (*i. e.* of that Extreme and Mean) which consists of the greatest Numbers is the greatest; so if $a:b$ are greatest, $b-a \urcorner d-c$.

Example:

$a : b :: c : d$
$4 : 6 :: 2 : 3$
$4 : 2 :: 6 : 3$

Demonstr. $b-a:d-c::a:c$; but by Supposition $a \urcorner c$, therefore $b-a \urcorner d-c$; or if a, c are greatest, then, because $c-a: d-b::a:b$, and $a \urcorner b$, therefore $c-a \urcorner d-b$.

Coroll. Of Numbers in $\div l$, $a:b:c:d$, &c. the Differences taken from the lesser Extreme a, do continually increase; thus, $b-a \angle c-b \angle d-c$, &c. for $a:b::b:c$, and b, c the two greatest, hence $c-b \urcorner b-a$, and so on, through the Series.

Scholium. In the last *Chapter* (*Theor.* 18.) it is demonstrated, that these Differeences make a Series $\div l$.

Theorem XIV.

Of four Numbers $::l$ the Sum of the Extremes and Means are unequal; and that Sum whose Parts are the least and greatest of the four, is the greatest.

$a : b :: c : d$
$3 : 2 :: 6 : 4$
$2 : 3 :: 4 : 6$

Demonstr. If a is the least, then is d the greatest (*Lemma*), and I say, $a+d \urcorner b+c$; for c, d or d, b are the two greatest; suppose c, d, then $d-c \urcorner b-a$ (*Theor.* 13.); add $a+c$ to both, and $d-c+a+c$ $(=d+a)$ is $\urcorner b-a+a+c$ $(=b+c)$. Or if d, b are the two greatest, then also $d-b \urcorner c-a$; add $a+b$ to each, and $d+a \urcorner b+c$.

If b is the least, then is c the greatest, and $b+c \urcorner a+d$; which is plain from the Alternation of the Terms, *viz.* $b:a::d:c$, whereby the Extremes are the least and greatest, as before.

Coroll. If three Numbers are $\div l$, the Sum of the Extremes exceeds double of the Means; thus, if $a:b::b:c$, then $a+c \urcorner 2b$. And, particularly, it exceeds by the Product of the lesser Extreme into the Square of the Difference betwixt the Ratio and 1, for if we take $a:ar:ar^2$, then is $a+ar^2=a\times\overline{r^2+1}$, and $2ar=a\times 2r$, also $a\times\overline{r^2+1}-a\times 2r=a\times\overline{r^2+1-2r}=a\times\overline{r-1}|^2$

Theorem XV.

If three Numbers, a, b, c, are $\div l$, and other three, d, e, f, also $\div l$, and in the same Ratio, *viz.* $a:b::d:e$, then the Ratio of the Sum of the Extremes to double the Means, is the same in both Ranks.

a, b, c
$4, 6, 9$
d, e, f
$8, 12, 18$
$a+c:2b::d+f:2e$
$4+9:2\times6::8+18:2\times12$
$13 : 12 :: 26 : 24$

Demonstr. $a:c::d:f$ (*Theor.* 19. *Chap.* 3.); hence $a+c:d+f::a:d$: But since $a:b::d:e$, therefore $a:d::b:e$; consequently $a+c:d+f::b:e::2b:2e$.

Or, *alternately*, $a+c:2b::d+f:2e$.

Theo-

$A : B : C$
$\overline{AB}^{\frac{1}{2}}$ $\overline{BC}^{\frac{1}{2}}$

Demonstr. Let $A:B:C$ be $\div l$, and the two Geometrical Means as in the Margin, *viz.* $\overline{AB}^{\frac{1}{2}}$ betwixt A and B, $\overline{BC}^{\frac{1}{2}}$ betwixt B and C; then the Squares of these two, with the Square of the Mean $\div l$, B, are $\div l$; for these Squares are $AB:BB:BC$, which are the Products of the Series $A.B.C$ by B, and consequently they are $\div l$ (*Th.* 3. *Ch.* 2.)

Theorem XXVIII.

It's possible to find three Numbers $\div l$, such, that the greater shall be equal to the Sum of the two lesser, but to find such three Numbers $\div l$ is impossible.

Demonstr. For the 1st Part, it's evident, and the Rule is this; Let the lesser Extreme be equal to the Difference, thus, $a.2a.3a$, where $3a=2a+a$. For the 2d Part, it's demonstrated thus; Let a, b, c be any three Numbers $\div l$; then a being the greater Extreme, and if $a=b+c$, then $b+c:b:c$, are $\div l$, whence $bc+cc=bb$; and if this is possible, then (by *Problem* 6. *Chap.* 2. *Book* III.) $c=\sqrt{bb+\frac{bb}{4}}-\frac{c}{2}$, or $\sqrt{\frac{5bb}{4}}-\frac{c}{2}$; but this is impossible, $\frac{5bb}{4}$ not being a Square Number; for tho' bb is a Square, yet 5 not being a Square, $5bb$ cannot be a Square (*Theor.* 2. *Cor.* 4. *Ch.* 1. *B.* III.) wherefore it's impossible that $b+c:b:c$ should be $\div l$.

Coroll. If three Numbers are $\div l$, it's impossible that the lesser Extreme should be equal to the Difference of the greater and Mean: For, in the preceding *Example* let $a-b=c$, then is $a=b+c$, which is inconsistent with $a:b:c$ being $\div l$.

Theorem XXIX.

If the Extremes of two Series $\div l$ having an equal Number of Terms, are $::l$ (or in the same Ratio), any two corresponding Terms in the one and other, are also $::l$.

$a . b . c . d . e$
$A . B . C . D . E$

Demonstr. If $a:e::A:E$, then $a:b::A:B$; for the two Differences are $b-a=\frac{e-a}{n-1}$, and $B-A=\frac{E-A}{n-1}$ (n being the Number of Terms, *Problem* 5. *Chap.* 2.); hence $b=a+\frac{e-a}{n-1}$, and $B=A+\frac{E-a}{n-1}$; but since $a:A::e:E$, therefore $a.A::e-a:E-A::\frac{e-a}{n-1}:\frac{E-A}{n-1}$, and hence again, $a:A::a+\frac{e-a}{n-1}:A+\frac{E-A}{n-1}$, or $b:B$; wherefore, lastly, $a:b::A:B$.

Again; Since $a:A::e:E$: and $a:A::b:B$, therefore $b:B::e:E$, or $b:e::B:E$; and because we may now consider b, e and B, E as Extremes, therefore, by the same Reasoning as before, it will follow that $b:c::B:C$, and so on, thro' the whole Series. And again; Taking any two corresponding Terms in each, at whatever Distance, they will be $::l$; because they are the Compound of the same simple Ratios; so $a:d::A:D$, also $b:d::B:D$, and so of others.

Theorem XXX.

Betwixt each Term of a Series $\div l$, take a Mean $: l$, and these Means are also $\div l$.

Demonstr. If $a . b . c . d . e.$ &c. are $\div l$, then the $\div l$ Means are $\frac{a+b}{2} : \frac{b+c}{2} : \frac{c+d}{2}$, &c. But (by *Theor.* 18. *Chap.* 3.) $a+b : b+c : c+d$, &c. are $\div l$, therefore their Halfs are so (*Gen. Coroll.* 15. *Chap.* 1.).

Theorem XXXI.

If three Numbers are $\div l$, and if to any one of the Extremes be added that Number which is a 3d $\div l$ to the other Extreme and the Difference, the Sum is a 3d $\div l$ to the preceding two Terms.

Example: $4 . 6 . 8$ are $\div l$, and $4 . 6 . 9$ are $\div l$, $9 = 8 + 1$, and 1 the 3d $\div l$ to $4, 2$, and 2 the common Difference in the Series $4 . 6 . 8$. Again; $8 . 6 . 4\frac{1}{2}$ are $\div l$, and $\frac{1}{2}$ the 3d $\div l$ to $8, 2$.

Demonstr. $a . a+b, a+2b$ are $\div l$. Let a, b, c be $\div l$, then are $a : a+b : a+2b+c, \div l$: For $a \times \overline{a+2b+c} = a^2+2ab+ac, = a^2+2ab+b^2$ (because $ac = b^2$, since $a : b : c$ are $\div l$) $= \overline{a+b}^2$, therefore $a : a+b : a+2b+c$ are $\div l$, the Product of the Extremes being equal to the Square of the Mean.

Or if we take these $\div l, a, a-b, a-2b$, and these, $a : b : c, \div l$, then $a : a-b : a-2b+c$ are $\div l$, for $a \times \overline{a-2b+c} = a^2-2ab+ac = a^2-2ab+b^2 = \overline{a-b}^2$.

Coroll. If three Numbers are $\div l$, $a : b : c$, then any one of the Extremes, a, the Sum of that Extreme and the middle Term, $a+b$, and the Sum of both Extremes with double of the middle Term, $a+2b+c$, make a Series $\div l$.

Problem I.

To find three Numbers $\div l$ such that the Quote (or Ratio) of the Ratio of the Extremes, and the Ratio of the Mean and common Difference, shall be greater than any assigned Ratio (taking the Ratios of the greater to the lesser).

Rule. Suppose the given Ratio is $d : a$, where $d \urcorner a$, then the three Numbers sought are $a : a+d : a+2d$; for the Ratio of the Extremes is $\frac{a+2d}{a}$, that of the Mean and Difference is $\frac{a+d}{d}$; then $\frac{a+2d}{a} \div \frac{a+d}{d} = \frac{da+2d^2}{a^2+da} = \frac{a+2d}{a+d} \times \frac{d}{a}$, which is greater than $\frac{d}{a}$.

Problem II.

To find a given Number of Terms $\div l$, whose Extremes are in the Ratio of two given Numbers, and the common Difference is the Difference of the same two Numbers.

Rule.

RULE. Suppose the given Numbers are a, b, $(a \angle b)$ and n the Number of Terms; these Products, *viz.* $\overline{n-1} \times a$, and $\overline{n-1} \times b$, are the Extremes of the Series sought; by which, with the given Difference, $b-a$, the Series may be filled up.

Example: The given Ratio 3 : 5, and Number of Terms 7, the Series is 18 . 20 . 22 . 24 . 26 . 28 . 30.

DEMONSTR. $a : b :: a \times \overline{n-1} : b \times \overline{n-1}$, and these, $a \times \overline{n-1}$, and $b \times \overline{n-1}$, being made the Extremes of a Series $\div l$, whose Number of Terms is n, the common Difference is $\frac{b \times \overline{n-1} - a \times \overline{n-1}}{n-1}$ (by *Probl.* 5. *Chap.* 2.) $= b - a$, dividing the Numerator and Denominator equally by $n-1$.

PROBLEM III.

To find two Series $\div l$ in which the Extremes are in two given Ratios, and the common Difference equal in both.

RULE. Suppose the given Ratios are $a : b$ and $c : d$, find (by the last) two Series whose Extremes are (the one) in the Ratio of $a : b$, and (the other) of $c : d$, with any proposed Number of Terms in each; then multiply each of these Series by the common Difference of the other.

Example: The Ratios 2 : 3, and 4 : 7. I first find these two Series, 6 . 7 . 8 . 9, whose Extremes are 6 : 9 (:: 2 : 3) and 12 . 15 . 18 . 21, whose Extremes are 12 : 21 (:: 4 : 7); then multiplying the 1st Series by 3, and the other by 1, the Products are 18 . 21 . 24 . 27, and 12 . 15 . 18 . 21 the Series sought.

DEMONSTR. All that needs Demonstration here is, that the two Series first found being multiplied by one another's Differences, the Products are two Series having the same Differences; which is plain (from *Theor.* 3. *Chap.* 2.); for let the Differences of the first two Series be d, c, then the Series whose Difference is d being multiplied by c, the Products are in the Difference $d \times c$; and the Series whose Difference is c being multiplied by d, the Products are in the Difference $c \times d$; but $d \times c = c \times d$, therefore the *Rule* is true.

PROBLEM IV.

To find a Series of Numbers $\div l$, whose Extremes are in a Ratio not less than a given one, and the Difference of the lesser Extreme and the Term next it not less than a given Number, the Number of Terms being also given.

RULE. Suppose the given Ratio $a : b$, and the given Number n: Then I find (by *Probl.* 2.) a Series $\div l$ of the Number of Terms proposed, whose Extremes are in the Ratio $a : b$, and whose Difference is $b-a$; if $b - a \angle n$, then I multiply the Series found by such a Number as shall make the Difference of the lesser Extreme and Term next it, at least $= n$. Take this Series, or the former Series if $b-a$ is $=$ or $\urcorner n$, and continue the two first Terms into a Series $\div l$ to the proposed Number of Terms; it is the Series sought.

A . B . C . D . E . F, &c. X.
A : B : L : M : N : O, &c Y.

DEMONSTR. Let A . B . C . D, &c. be a Series $\div l$ of the given Number of Terms, whose Extremes, A : X, are in the given Ratio, $a : b$, and B − A ($= b - a$) not less than n: Then the 2d Series, A . B . L . M, &c. being $\div l$, of the same

ſame Number of Terms, and the 1ſt and 2d Terms being the ſame, two Conditions of the *Problem* are anſwered, *viz.* the Number of Terms, and Difference of the 1ſt and 2d Terms: What remains to be ſhewn then is only this, that the Ratio of the Extremes Y : A is not leſs than the given Ratio $b : a$, or X : A; which is thus proved: In the Series $\div l$, A : B : C, *&c.* the Ratios of every Term to the preceding leſſer Term do conſtantly decreaſe (*Theor.* 16), and X : A is in the compound Ratio of all the intermediate Ratios; but in the Series $\div l$, A : B : L, *&c.* the Ratios being equal, and Y : A in the compound Ratio of the intermediate ones, it muſt be greater than X : A. Or thus; In the Geometrical Series the Differences from the leſſer Extreme do conſtantly increaſe (by *Theor* 13), but in the Series $\div l$ they are the ſame; therefore Y — A is greater than X — A, and Y $\urcorner$ X, conſequenly Y : A $\urcorner$ X : A.

Scholium. If the Problem is propoſed ſo that the Extremes of the Series to be found ſhall be in a Ratio not exceeding the given one, then proceed thus: Having found a Series $\div l$, whoſe Extremes are in the given Ratio $a : b$, and the Difference of the leſſer Extreme and that next it $= b - a$, as that repreſented in the Margin, A B. C, *&c* U . X, wherein X : A : : $b : a$, and the common Difference $= b - a$; then I contrive a Series $\div l$, downwards from X : U to as many Terms as the other Series, as X : U, *&c.* N : M : L; and becauſe the Differences decreaſe from the greater Extreme X, therefore M — L is leſs than X—U; for the ſame Reaſon, and the Equality of the Differences in the other Series, it's plain that L is greater than A, and conſequently X : L is leſs than X : A (or $b : a$, the given Ratio): But, laſtly, becauſe M — L $\angle$ $b - a$, I multiply the whole Series by ſuch a Number as makes nM — nL not leſs than $b - a$; and ſo theſe Products make the Series ſought.

A .	B .	C,	*&c.*	U .	X
L .	M .	N,	*&c.*	U .	X
nL .	nM .	nN,	*&c.*	nU .	nX

CHAP.

THEOREM XVI.

If three Numbers, a, b, c, are $\div l$ (a the greateſt), and other three, d, e, f, alſo $\div l$ (d the greateſt), but in a leſſer Ratio than the other; then the Ratio of the Sum of the Extremes to double the Mean in the 1ſt Rank is greater than that in the other; thus, $a+c : 2b \urcorner d+f : 2e$.

a, b, c
d, e, f
l, b, m

DEMONSTR. Take three Numbers, $l, b, m, \div l$, whoſe Mean is the ſame as that in the 1ſt Rank, and the Ratio equal to that in the 2d Rank, *viz* $d:e::l:b$; then, becauſe $a:b \urcorner d:e$, therefore $a:b \urcorner l:b$, conſequently, $a \urcorner l$; and becauſe alſo $b:c \urcorner b:m$, therefore $c \angle m$; but $ac = bb = lm$, hence $a:l::m:c$; and becauſe a is the greateſt, and c the leaſt of the four, therefore $a+c \urcorner l+m$ (*Theor.* 14.), and conſequently, $a+c:2b \urcorner l+m:2b$, which proves the *Theorem* in this Caſe. But, again (by laſt *Theor.*), ſince d, e, f, and l, b, m, are $\div l$ in the ſame Ratio, therefore $d+f:2e::l+m:2b$; and it's now ſhewn, that $a+c: 2b \urcorner l+m:2b$, therefore $a+c:2b \urcorner d+f:2e$.

THEOREM XVII.

Of four Numbers $:l$, the Ratio of that Couplet which conſiſts of the greater Numbers is greateſt, comparing the leſſer Term to the greater, and, contrarily, comparing the greater to the leſſer.

$a, b : c, d$
$5, 7 : 8, 10$

DEMONSTR. If a, b, c, d are $:l$, and c, d the two greater, alſo $d \urcorner c$, then $\frac{c}{d} \urcorner \frac{a}{b}$; for let $d = c+n$, and $b = a+n$, then $a.\ a+n:c.\ c+n$. Take their Ratios, *viz.* $\frac{a}{a+n}$ and $\frac{c}{c+n}$, and compare them by reducing them to one common Denominator, the new Numerators are $ac+an$, and $ac+nc$, which is greater than the former, becauſe ac is common to both, and c being $\urcorner a$, $nc \urcorner na$, conſequently, $\frac{c}{c+n}\left(=\frac{c}{d}\right)$ is $\urcorner \frac{a}{a+n}\left(=\frac{a}{b}\right)$.

COROLL. In a Progreſſion $\div l$, the Ratios of every Term to the next from the leſſer Extreme do continually increaſe; thus, if a, b, c, d, &c. are $\div l$, and a the leſſer Extreme, then is $\frac{a}{b} \angle \frac{b}{c} \angle \frac{c}{d}$ &c. *Example:* Of this Progreſſion, 1, 2, 3, 4, 5, &c. the Ratios increaſe $\frac{1}{2} \angle \frac{2}{3} \angle \frac{3}{4} \angle \frac{4}{5}$, &c.

THEOREM XVIII.

Of four Numbers $:l$, the Product of the Extremes and Means are unequal, and that of the leaſt and greateſt Terms is the leaſt Product.

$a, b : c, d$
$3.\ 7 : 5.\ 9$

DEMONSTR. Suppoſe a the leaſt and d the greateſt, then either $c \urcorner$ or $\angle$ or $= b$. Suppoſe $c \urcorner$ or $= b$, then are c, d, the two greater Terms, and, by the laſt, $\frac{c}{d} \urcorner \frac{a}{b}$, and, conſequently $cb \urcorner ad$. Again; Suppoſe $c \angle b$, then are b, d the two greateſt, and if we alternate the Terms thus, $a.c: b.d$, then, by the laſt, $\frac{b}{d} \urcorner \frac{a}{c}$, and hence $bc \urcorner ad$.

If the leaſt and greateſt are the two Means, then, by reverſing the Terms, they become the Extremes, and then it falls within the preceding *Demonſtration.*

COROLL.

COROLL. If three Numbers are $\div l$, $a.b.c$, the Product of the Extremes, a, c, is less than the Square of the Mean bb, and that by the Square of the Difference; thus, Take $a.a+d,\ a+2d$, and $a\times\overline{a+2d}=a^2+2ad$; also, $\overline{a+d}^2=a^2+2ad+d^2$.

THEOREM XIX.

IF there are three Numbers, $a, b, c, \div l$, and other three, d, e, f, also $\div l$, and with the same Difference, the Product of the Extremes, and Square of the Means have the same Difference in both Ranks; thus, $ac-bb=df-ee$.

a, b, c
d, e, f

DEMONSTR. Let the common Difference in the two Ranks be n, then (by *Theor.* 18.) $ac-bb=nn$; also $df-ee=nn$, therefore $ac-bb=df-ee$.

THEOREM XX.

IF three Numbers, a, b, c, are $\div l$, and other three, d, e, f, also $\div l$, but with a lesser Difference, the Difference of the Product of the Extremes, and the Square of the Mean in the 1st Rank, is greater than that in the 2d Rank.

a, b, c
d, e, f

DEMONSTR. Let $b-a=m$, and $d-e=n$, then (by *Theor.* 18.) $ac-bb=m^2$, and $df-ee=n^2$; but, by Supposition, $m \urcorner n$, hence $m^2 \urcorner n^2$, and consequently $ac-bb \urcorner df-ee$.

THEOREM XXI.

THE Mean Arithmetical betwixt two Numbers is a greater Number than the Mean Geometrical.

DEMONSTR. The Product of the Extremes is equal to the Square of the Mean Geometrical, but it's less than the Square of the Mean Arithmetical (by *Cor.* to *Theor.* 18.) consequently this Mean is greater than the other: Thus, betwixt A, B, let the Mean $\div l$ be C, and the Mean $\dot{\div} l$ D; then is $AB=DD$, but $AB \angle CC$, therefore $DD \angle CC$, or $CC \urcorner DD$. *Example*: Betwixt 2 and 8 the Mean $:l$ is 5, and the Mean $::l$ is 4.

SCHOLIUM. If we express these Means according to their proper Rules, we find the Mean $:l$ exceeds the Mean $::l$, by half of the Difference betwixt the Sum of the Extremes, and double of the Geometrical Mean; for these are $\dot{\div} l$, $a:ar:arr$, and the Arithmetical Mean betwixt a and arr is $\frac{a+arr}{2}$: And, lastly, $\frac{a+arr}{2}-ar=\frac{a+arr-2ar}{2}$.

THEOREM XXII.

TO the same three Numbers, a, b, c, take a fourth $:l$, n, and a 4th $::l$, d, and order them so that the two first Terms, a, b, are either the two least or the two greatest of the four Terms, then is $n \urcorner$ or $\angle d$ in these different Circumstances, *viz.* (1.) If a, b are the two least, and $a \urcorner b$, then is $n \urcorner d$. (2.) If a, b are the two least, and $a \angle b$, then is $n \angle d$. (3.) If a, b are the two greater, and $a \urcorner b$, then is $n \angle d$. (4.) If a, b are the two greater, and $a \angle b$, then is $n \urcorner d$.

DEMON-

$a \,.\, b : c \,.\, n$
$a : b :: c : d$

DEMONSTR. 1. If a, b are the two leſſer, and $a \urcorner b$, then is $\frac{a}{b} \urcorner \frac{c}{n}$ (*Theor.* 17.) But $\frac{a}{b} = \frac{c}{d}$, hence $\frac{c}{d} \urcorner \frac{c}{n}$, and conſequently $n \urcorner d$. (2.) a, b, the two leſſer, and $a \angle b$, then is $\frac{a}{b} \angle \frac{c}{n}$ (*Theor.* 17.); but $\frac{a}{b} = \frac{c}{d}$, hence $\frac{c}{d} \angle \frac{c}{n}$, and $n \angle d$. (3.) a, b the two greateſt, and $a \urcorner b$, $\frac{a}{b} \angle \frac{c}{n}$ (*Theorem* 17.); but $\frac{a}{b} = \frac{c}{d}$, hence $\frac{c}{d} \angle \frac{c}{n}$, and $n \angle a$. (4.) a, b the two greater, and $a \angle b$, then is $\frac{a}{b} \urcorner \frac{c}{n}$; hence $\frac{c}{d} \urcorner \frac{c}{n}$, and $n \urcorner a$.

COROLL. To two Numbers a third $\div l$ increaſing, is leſs than a third $:: l$; but, decreaſing it is greater.

THEOREM XXIII.

OF four Numbers $: l$, take their Squares: They cannot poſſibly be $: l$, but the Difference of the 1ſt and 2d Square will be to the Difference of the 3d and 4th in the Ratio of the Sums of the common Difference, and double the leſſer Term in the reſpective Couplets. Thus, if $a \,.\, b : c \,.\, d$, then $b^2 - a^2 : d^2 - c^2 :: 2a + \overline{b-a} : 2c + \overline{d-c}$.

DEMONSTR. Take four Terms, thus, $a \,.\, a+d : b \,.\, b+d$, their Squares are a^2, $a^2+2ad+d^2$, b^2, $b^2+2bd+d^2$, and the Differences are $2ad+d^2$, $2bd+d^2$, which, by equal Diviſion, are as $2a+d : 2b+d$.

COROLL. The Squares of a Series $: l$ cannot poſſibly be $: l$.

THEOREM XXIV.

OF four Numbers $: l$, the Difference betwixt the Sum of the Squares of the Extremes and the Sum of the Squares of the Means, is equal to the Product of twice the common Difference, multiplied into the Difference of the 1ſt and 3d Terms, the Terms being ſo ordered that the 1ſt and 3d are leſs than the 2d and 4th. Thus, take $a \,.\, b : c \,.\, d$, where $a \angle b$, and $c \angle d$, then $a^2 + d^2 - b^2 + c^2 = \overline{c-a} \times 2 \times \overline{b-a}$.

DEMONSTR. Take $a \,.\, a+d : b \,.\, b+d$, their Squares are a^2, $a^2+2ad+d^2$, b^2, $b^2+2bd+d^2$; the Sum of the Squares of the Extremes is $a^2+b^2+2bd+d^2$, and of the Means it is $b^2+a^2+2ad+d^2$; and the Difference of theſe Sums is plainly the Difference of $2bd$ and $2ad$ (the other Parts being equal), *i.e.* $2bd - 2ad = 2d \times \overline{b-a}$, or $2ad - 2bd = 2d \times \overline{a-b}$.

COROLL. If three Numbers, a, b, d, are $\div l$, or $2b = a+d$, then the Difference of the Sum of the Squares of the Extremes and double the Square of the Mean, is double the Square of the common Difference.

THEO-

Theorem XXV.

If three Numbers are $\div l$, the Ratio of the Extremes cannot possibly be the same as that of the Mean and common Difference.

Demonstr. Take $a . a+d . a+2d$; I say, it's impossible that it can be $a : a+2d :: d : a+d$; for if it is, then $a \times \overline{a+d} = d \times \overline{a+2d}$, i. e. $a^2 + ad = ad + 2d^2$; whence, taking ad from both, it is $a^2 = 2d^2$, which is impossible since 2 is not a Square; for then a Square d^2, and a Number not a Square would produce a Square a^2, contrary to what is demonstrated in *Ch.* 1. *Book* III. See *Coroll.* after *Theor.* 2.

Scholium. If the Extremes are in the Ratio of $1 : 2$, the Difference and Mean will be as $1 : 3$, as this *Example* shews; $a . a+d . a+2d = 2a$, whence $a = 2d$, and $a+d = 3d$; and, reciprocally, if the Extremes are in the Ratio of $1 : 3$, the Difference and Mean are in the Ratio of $1 : 2$, as here, a, $2a$, $3a$. In *Theorem* 26. you'll find this Reciprocality demonstrated universally, whatever the Ratios are.

Theorem XXVI.

If there are two Ranks of three Numbers each, $\div l$, and if the Ratio of the Extremes in the one Rank is equal to the Ratio of the Mean and Difference of the other Rank, then, reciprocally, the Ratio of the Extremes in the last Rank is equal to the Ratio of the Mean and Difference in the former Rank.

a .	$a+d$.	$a+2d$
6 .	8.	10
d	$\frac{a+2d}{2}$	$a+d$
2	5	8

Demonstr. In the annex'd *Example*: The Extremes of the 2d Series are the Difference and Mean of the 1st Series, and therefore in this Case it remains only to be shewn that the Ratio of the Extremes of the 1st Series is that of the Mean and Difference of the 2d; which is plain, for $a+2d : a :: \frac{a+2d}{2} : \frac{a+2d}{2} - d = \frac{a+2d-2d}{2} = \frac{a}{2}$; but $\frac{a+2d}{2} : \frac{a}{2} :: a+2d : a$, therefore,

Again, Whatever three Numbers we take that are $\div l$, and whose Extremes are as $d : a+d$; for *Example*, D, E, F, where we suppose $D : F :: d : a+d$, then (by *Theor.* 31. below) $D : E :: d : \frac{a+2d}{2}$, and $E - D : E :: \frac{a+2d}{2} - d \left(= \frac{a}{2}\right) : \frac{a+2d}{2} :: a : a+2d$.

Theorem XXVII.

If three Numbers $\div l$ admit betwixt each of them a Geometrical Mean, the Square of the Mean $\div l$ is a Mean $\div l$ betwixt the Squares of the two Geometrical Means.

CHAP. VI.

Of Harmonical Proportion; in which I use this Sign, hl, *for the Words* Harmonically Proportional.

§. 1. *Contains the* THEORY, *considered purely as an* Arithmetical Doctrine.

PROBLEM I.

HAVING three Numbers, to find a fourth *hl* with them, taken in a certain given Order.

Rule. Take the Product of the first and third, divide it by the Difference of the second and double the first, the Quote is the Number sought. So to these, A, B, C, a fourth *hl* is $\frac{AC}{2A-B}$

Exa. To these 3, 4, 6, a fourth *hl* is 9; thus $3\times6=18$; then 2×3 $(=6)$ $-4=2$. Lastly, $18\div2=9$; so that these 4 are *hl*, *viz.* 3, 4, 6, 9, for these are geometrically Proportional, *viz.* $3:9::4-3\ (=1):9-6\ (=3)$

Demon. Let four Numbers *hl* be represented by these Letters, A, B, C, D; then if A is greater than B, these are :: *l*, *viz.* $A:D::A-B:C-D$. Hence $AC-AD=AD-BD$, and adding AD to both these Equals, it is $AC=2AD-BD=\overline{2A-B}\times D$, and dividing equally by $2A-B$, it is $D=\frac{AC}{2A-B}$, which is the Rule.

Or if A is less than B, then $A:D::B-A:D-C$; hence $AD-AC=BD-AD$, and adding AC to both, it is $AD=BD-AD+AC$. Again, substracting $BD-AD$ from both, it is $AC=2AD-BD$ $(=\overline{2A-B}\times D)$ and dividing by $2A-B$, it is $D=\frac{AC}{2A-B}$

COROLLARIES.

1*st*. We can by the same Method find a third *hl* to two given Numbers, if we take the second Term twice to make three given Numbers; and then the Rule will be plainly thus; divide the Product of the two given Numbers by the Difference betwixt the second and double the first, the Quote is the Number sought.

Exa. To these 3, 4, a third *hl* is 6; and to these 6, 4, it is 3; as you'll find by the Rule universally to these A, B, a third *hl* is $\frac{AB}{2A-B}$

SCHOLIUM (1°.) A third or 4th *hl* is always possible when 2A, (or double the first Term) is greater than the second B; but not otherways. The Reason is plain, because the Divisor $2A-B$ is then some real Number; but if 2A is less or only equal to B, then there is no real Divisor, so that a third or fourth *hl* to the given Numbers is impossible.

Exa. To these 3, 6, or 3, 6, 7, there is no third or fourth *hl*, because double the first Term is equal to the second; nor to these 2, 5, or 2, 5, 6, because double the first Term is less than the second.

2°. Observe also these other Characters of two or three Numbers, to which a third or fourth *hl* is possible, *viz.* If to two Numbers, a third arithmetically proportional (which two last Words are marked thus : *l*) can be found either increasing or decreasing (as it can always be increasing) then a third *hl* to the same two Numbers can be found contrarily decreasing or increasing; because in this Case double the first Term of the Harmonicals is always greater than the second. Thus if A, B, C, are : *l*, then 2 B = A + C; consequently a third *hl* to B, C is possible. Again, if a third : *l* to C, B, is possible, a fourth *hl* to B, C, and any other Number, as B, C, D, is also possible; for the same Reason, *viz.* because 2 B is greater than C. *Lastly*, If three Numbers are : : *l*, and if to the middle Term with either extreme, a third *hl* is possible; to the same three Numbers a fourth *hl* is possible in this Order, *viz.* If A, B, C, are : : *l*, and if to B, C, a third *hl* is possible, then also to A, B, C a fourth *hl* is possible; for since A : B : : B : C, hence 2 A : B : : 2 B : C; but 2 B is greater than C, else a third *hl* to B C would not be possible; therefore 2 A is greater than B, which makes a fourth *hl* to A, B, C possible.

The Reverse of all these are also true and necessary to be here remarked; thus, if to B, C, or B, C, D, a third or fourth *hl* is possible, then reversely, to C, B, a third : *l* as A is also possible; for by Supposition 2 B is greater than C, which is all the Condition necessary to make A possible, since A = 2 B — C. Again, if A, B, C, are : : *l*, and if to A, B, C, a fourth *hl* is possible, then a third *hl* to B C is also possible; for this requires only that 2 B be greater than C, which it will be when 2 A is greater than B, as it is in case a fourth *hl* to A, B, C, is possible.

2*d*. From any given Number a Progression or Series *hl* may be found decreasing in Infinitum, but not increasing; for it will stop whenever the last found Term is equal to or exceeds the Double of the preceding Term; so this *hl* Series 12, 15, 20, 30, can be continued no further increasing, because 30 is less than 40, = 2×20; yet it's possible to find two Numbers from which any assigned Numbers of Terms *hl* may proceed increasing, as you'll learn in *Theorem* second, or to find a Series *hl* of any Number of Terms, all Integers; which cannot be done from any given Number, though we take a Series decreasing, because 2 A — B will not be in every Case an aliquot Part of A B.

PROBLEM II.

To find a Mean hl *betwixt two Numbers.*

Rule. Divide double their Product by their Sum, the Quote is the Mean sought; thus, betwixt A, B, a Mean *hl* is $\frac{2\ A\ B}{A+B}$

Exa. Given 3, 6, the Mean is 4; thus, 3×6 = 18, then 2×18 = 36 and 36 ÷ 9 = 4. For these are *hl* 3, 4, 6, because 3 : 6 : : 4 — 3 (= 1) : : 6 — 4 (= 2.)

Demon. If A, B, C, are *hl*, decreasing from A, and increasing from C, then it is A : C : : A — B : B — C, and reversly C : A : : B — C : : A — B, whence A B — A C = A C — B C; and adding B C to each, it is A B — A C + B C = A C; again, adding A C to each, it is A B + B C ($= \overline{A+C} \times B$) = 2 A C; and, *Lastly*, B = $\frac{2\ A\ C}{A+C}$; which is the Rule.

THEOREM I.

If four Numbers *hl* be equally multiplied, or divided, the Products or Quotes are also *hl*.

Exa.

Exa. 1ſt. If theſe, 3, 4, 6, 9, are multiplied by 2, the Products 6, 8, 12, 18, are *hl*, which again divided by 2 Quote the former.

Demon. The Products or Quotes of the Extreams are ſtill in the ſame Ratio with the Extreams; and the Differences of the Products of the middle Term and Extreams, though they are greater or leſſer than the Differences of theſe Terms themſelves, yet are ſo proportionally, and that in the Ratio of the Multiplier or Diviſor, which is the Ratio of the Products of the Extreams; therefore there is ſtill a geometrical Proportion in the Products or Quotes, betwixt the Extreams and the Differences of theſe from the mean Terms, *i. e.* theſe Products or Quotes are *hl*.

Or, ſee this Truth alſo in univerſal Characters thus, 1°. Let theſe be *hl*, *viz.* A, B, C, D, that is A : D : : A—B : C—D; then are theſe *hl*, Ar, Br, Cr, Dr, That is Ar : Dr : : Ar—Br : Cr—Dr, for theſe are the plain Effects of multiplying *r* into the preceding : : *ls*, *viz.* A : D : : A—B : C—D, therefore the Products are : : *l*, *viz.* Ar : Dr : : Ar—Br : Cr—Dr, *That is*, Ar, Br, Cr, Dr, are *hl*, according to the Definition. As for Diviſion, it is but the Reverſe of the firſt, and ſo its Demonſtration is contained in it.

COROLLARIES.

1ſt. Any three Terms *hl* being equally multiplied or divided, the Products or Quotes will be *hl*; becauſe by taking the middle Term twice, they make four Terms *hl*, for 2, 3, 6, is the ſame as 2, 3, 3, 6, as to *hl*.

2d. As any three Numbers *hl*, ſo by equal Reaſon any Series continually *hl* being equally multiplied or divided, the Products or Quotes are ſtill *hl*.

3d. By this *Theorem* and *Problem* firſt, we learn how to find a Series *hl*, conſiſting of any Number of Terms, all Integers; thus, begin with any two Integers, and find a third *hl* decreaſing; if it's a Fraction, or mix'd Number reduced to a Fraction, then multiply the two given Numbers by the Denominator of this Fraction, the Products with the Numerator are three Terms *hl*. To theſe join another Term *hl* decreaſing; and if it's Fractional, multiply all the preceding by its Denominator, and theſe Products with the Numerator are four Terms *hl* continually; after this manner go on to any Number of Terms. But you'll learn an eaſier Way of ſolving this Problem afterwards.

4th. We learn here alſo, how to find a Series $\div$ *l* or $\div\div$ *l* in Integers, betwixt every two adjacent Terms of which, there falls a *hl* Mean alſo Integral. *Thus*, take any Series $\div$ *l*, or $\div\div$ *l* all Integers; then find the *hl* Means, which being all or part of them mix'd Numbers, reduce the whole Series firſt found, together with the Means, to a common Denominator, and the Numerators give the Numbers ſought, and alſo the Means.

Theorem II.

If there is a Series of Numbers, $\div$ *l*, as, *a*, *b*, *c*, *d*, &c. increaſing or decreaſing, their Reciprocals, *viz.* $\frac{1}{a}$ $\frac{1}{b}$ $\frac{1}{c}$ $\frac{1}{d}$, &c. is a *harm.* Progreſſion, contrarily decreaſing or increaſing; and reverſely, if *a*, *b*, *c*, *d*, &c. are *hl*. $\frac{1}{a}$, $\frac{1}{b}$, *&c.* are $\div$ *l*.

Demon. (1°) If *a*, *b*, *c*, *d*, &c. is a Series increaſing or decreaſing, their Reciprocals $\frac{1}{a}$, $\frac{1}{b}$, *&c.* muſt contrarily decreaſe or increaſe. This follows evidently from the Nature of Fractions (V. *Lemma* 5th, Chap. 1ſt, B. 2d.)

2°. If the Thing aſſerted in the *Theorem* be true of any three Numbers in Progreſſion $\div$ *l* or *hl*, it muſt neceſſarily be ſo, how many Terms ſoever be in the Progreſſion; becauſe for a Series to be continuedly : *l*, or *hl*, is no other Thing than to have every three Terms in the continued Order of the Series : *l* or *hl*.

3°. What remains to be proved then is only this, *viz.* That if any three Terms, a, b, c, are $:l$ or hl, their Reciprocals must be contrarily hl: or $:l$; which I shew thus, (1°) Suppose a, b, c, are $\div l$, then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are hl, that is, $\frac{1}{a} : \frac{1}{c} :: \frac{1}{a} - \frac{1}{b} \left(= \frac{b-a}{ab}\right)$: $\frac{1}{b} - \frac{1}{c} \left(= \frac{c-b}{bc}\right)$; for since a, b, c, are $\div l$, then $b-a = c-b$, and hence $\frac{c-b}{abc} = \frac{b-a}{abc}$. That is, $\frac{1}{a} \times \frac{c-b}{bc} = \frac{1}{c} \times \frac{b-a}{ab}$, whence it is, $\frac{1}{a} : \frac{1}{c} :: \frac{b-a}{ab} : \frac{c-b}{bc}$. 2°. Suppose a, b, c, are hl, then, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are $\div l$, for since a, b, c, are hl, i. e. $a : c :: b-a : c-b$. Therefore $\frac{b-a}{a} = \frac{c-b}{c}$, and dividing equally by b it is $\frac{b-a}{ab} = \frac{c-b}{cb}$. But $\frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}$, and $\frac{c-b}{cb} = \frac{1}{b}\ \frac{1}{c}$ hence $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are $\div l$.

Exam. 1. 2, 3, 4, are $\div l$, and $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, hl; for $\frac{1}{2} : \frac{1}{4} :: \frac{1}{2} - \frac{1}{3} \left(= \frac{1}{6}\right) : \frac{1}{3} - \frac{1}{4} \left(= \frac{1}{12}\right)$
Exam. 2. 3, 4, 6, are hl, and $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, are $\div l$, for $\frac{1}{3} - \frac{1}{4} \left(= \frac{1}{12}\right) = \frac{1}{4} - \frac{1}{6} \left(= \frac{1}{12}.\right)$

SCHOLIUM.

1*st*. This Truth is universal, whether the supposed Numbers are Integers or Fractions; because $\frac{1}{a}$ expresses the Reciprocal of a, whatever a be, from the Nature of Division. If a is Integer the Thing is plain. If $a = \frac{n}{m}$, then is $1 \div a = 1 \div \frac{n}{m} = \frac{m}{n}$ the Reciprocal of $\frac{n}{m} \left(= a.\right)$

2*d*. If any Series consists of Fractions (either all, or only some of the Terms) then if the whole Series is reduced to a common Denominator, the new Numerator is a Series of Integers of the same Kind of Progression ($\div l$, hl or $\div\div l$) with that reduced; because the new Fractions are so, since they are equal to the former ones, and the Numerators are Equimultiples of the Fractions (for they are their Multiples by the common Denominator, since $\frac{a}{b} \times b = a$;) and, *Lastly*, the Equimultiples of any Series, $\div l$, $\div\div l$ or hl, are of the same Kind; as has been shewn of each in their Places.

3*d*. If any Series consists all of Integers, the most convenient way of reducing the Series of their Reciprocals (which are all Fractions) to a common Denominator, is this,

Arithmeticals,	2,	3,	4,	5
	3,	2		
		4,	3	
Harmonicals,	12,	8,	6	
			5,	4
Harmonicals,	60,	40,	30,	24

Of the given Series of Integers take every Couplet reversely in Order from the beginning, and continue them as so many Geometrical Ratio's (by *Probl.* I. *Chap.* 4.) as in the annex'd Example; the Operation of which you'll easily perceive to be the same as that which finds the new Numerators where the Reciprocals of these Numbers are reduced to a common Denominator; thus, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, reduced, the new Numerators are $3 \times 4 = 12$; $2 \times 4 = 8$, $2 \times 3 = 6$. And $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, reduced, the Numerators are $3 \times 4 \times 5 = 60$; $2 \times 4 \times 5 = 40$; $2 \times 3 \times 5 = 30$; $2 \times 3 \times 4 = 24$, and so on; which is yet more evident in this Example in Letters. But yet again,

4°. With-

a,	*b*,	*c*,	*d*
b,	*a*		
	c,	*b*	
bc,	*ac*,	*ab*,	
		d,	*c*
bcd,	*acd*,	*abd*,	*abc*

4°. Without regard to the Rule for reducing Fractions to one Denominator, we can easily demonstrate that these Products have the Quality asserted in the *Theorem*, *viz.* that if *a*, *b*, *c*, *d*, &c. are $\div l$ or *hl*, the Products so made are contrarily *hl* or $\div l$; *Thus*, these Products are Series in the reciprocal Ratio's of the given Series (as has been demonstrated in the *Problem* referr'd to) but the Reciprocals of the given Series are Numbers also in the reciprocal Ratio's of the same Series, therefore both these Series (*viz.* the Series of Products and of Reciprocals) have the same Quality; which is thus proved. Let *a*, *b*, *c* represent the Reciprocals of any three Numbers, and A, B, C, other three Numbers, in the same Ratio's, *viz.* $a : b :: A : B$ and $b : c :: B : C$, then comparing these, it is $a : A :: c : C$ or $a : c :: A : C$. Also, $a : A :: b-a : B-A$, and $c : C :: c-b : C-B$. Again, comparing the last three Proportions, it is $b-a : B-A :: c-b : C-B$; or $b-a : c-b :: B-A : C-B$. These Preparations being made, suppose now that *a*, *b*, *c*, are *hl*, *i. e.* $a : c :: b-a : c-b$, then are A, B, C, *hl*; or, $A : C :: B-A : C-B$; for because $a : c :: A : C$, therefore $A : C :: b-a : c-b$, and because, $b-a : c-b :: B-A : C-B$, hence A, B, C, are *hl*: Again, suppose *a*, *b*, *c*, are $\div l$, or $b-a=c-b$, so are A, B, C, or, $B-A=C-B$. For it is above shewn that $b-a : c-b :: B-A : C-B$, and therefore if $b-a=c-b$ so must $B-A=C-B$. In the last Place, since the Thing is true of three Terms, it's true of any Number in a Progression.

COROLLARIES.

1*st.* Here then we learn a very easy Way of finding a Series *hl*, consisting of any Number of Terms, all Integers; *thus*, take any Series : *l* of the same Number of Terms, all integral, and multiply their Terms together, according to the Direction in the second *Scholium*, and you have a Series *hl* of as many Terms, all Integers.

2*d.* If it were required to assign two Numbers, to which any Number of Terms *hl* are possible, increasing, without finding the whole Series; it's done by taking any Series : *l* consisting of the given Number of Terms, and multiplying continually into one another all the Terms of the Series : *l*, except the greater Extreme; for their Product is the lesser of the Numbers sought, and the continual Product of all the Terms, except that next the greatest Extreme, is the greater of the two Numbers sought.

But if it's also required that both the Terms sought, and also all that are proposed as possible to be added to them, be Integers; then let the assumed Series : *l* be all Integers.

3*d.* Further, though a Series *hl* may be continued infinitely decreasing from any given Number; yet the Terms will not all be Integers; therefore if it's required to find two Numbers, to which a given Number of Terms *hl* may be found decreasing, and all Integers, its done *thus*; take a Series : *l*, consisting of the given Number of Terms, all Integers, and the continual Product of all the Terms, except the lesser Extreme, is the greater of the Numbers sought, and the continual Product of all, except the Term next the lesser Extreme, is the lesser of the Terms sought. All this is plain from the preceding Operation, wherein every Term of the Harmonicals is the continual Product of all the Terms of the Arithmeticals, except the Correspondent in order, *i. e.* the 1*st*, 2*d*, or 3*d*, &c. if it is the 1*st*, 2*d*, or 3*d*, of the Harmonicals.

4*th.* If any Numbers, *a*, *b*, *c*, &c. which are $\div l$ or *hl*, are applied as Divisors to the same Number, N, the Quotes are contrarily *hl* or $\div l$; because Quotes are in the reciprocal Ratio's of the Divisors when the Dividend is common.

5th. Hence we learn the Solution of this Problem, *viz.* to find a Progression of any Number of Terms *hl*, and whose Extremes shall be in any assigned Ratio; which is solved *thus:* Find a Series ÷*l* of the same Number of Terms, and whose Extremes are in the given Ratio (by *Probl.* II. Ch. 5.) then the Reciprocals of these or their Numerators, being all reduced to a common Denominator, are the Numbers here sought.

6th. We have also the Solution of this Problem, *viz.* to find the first of any Number of Harmonical Means betwixt two given Numbers, *thus*, Take the Reciprocals of the given Numbers, and find the first of the proposed Number of ÷*l* Means betwixt them (as directed in Article 2*d* of the *Schol.* after *Coroll.* 8. *Probl.* 4. Ch. 5.) the Reciprocal of this Mean is the *hl* Mean sought.

7th. We have also the Solution of this Problem, *viz.* to find two Integers, betwixt which a proposed Number of *hl* Means, all Integers, may be found; *Thus*, take any Series ÷*l* betwixt whose Extremes there is the proposed Number of Means; the continual Product of all the Antecedents, and the continual Product of all the Consequents are two Numbers such as required; this is manifest from *Scholium* 3*d*.

THEOREM III.

If four Numbers are : : *l*, A : B : : C : D; and if betwixt each Couplet you take a *hl* : Mean, as *x* betwixt A : B, and *y* betwixt C : D; then the two Antecedents or two Consequents with these two *hl* Means, are also : : *l*; thus A : *x* : : C : *y*, and B : *x* : : D : *y*.

Demon. Let it be A : B : : Ar : Br. (instead of C, D) then A, *x*, B, being multiplied by *r*, the Products Ar, xr, Br, are *hl* (*Theo.* I.) also A : *x* : : Ar : xr. But Ar=C and Br=D, therefore xr=y. conseq. A : *x* : : C : *y*. But again, A : C : : B : D, therefore, *Lastly*, B : *x* : : D : *y*.

COROLL. If betwixt every two adjacent Terms of a Geometrical Series a *hl* Mean is taken, these Means make also a Geometrical Series in the same Ratio.

Geom. *a*, *b*, *c*, *d*	For *a* : *l* : : *b* : *m*; or *a* : *b* : : *l* : *m*; also, *b* : *c* : : *m* : *n*,
Hl. Means *l*. *m*, *n*.	therefore *l* : *m* : : *m* : *n*; and so on.

THEOREM IV.

In an Harmonical Progression, any three Terms, whereof the Middle is equally distant from the Extremes, are also *hl*.

Exam. In this Series 10, 12, 15, 20, 30, 60, these three are *hl*, 10, 15, 30; also these 12, 20, 60.

Demon. Since by *Schol.* 3d. to *Theor.* II. an Arithmetical Series can be found, each immediate Couplet whereof is in the reciprocal Ratio's of the Correspondent Harmonicals, and of the Series : *l*, any three Terms, whereof the Middle is equally distant from the Extremes, are : *l*; therefore the Correspondent to these in the Series *hl* must also be *hl*.

THEOREM V.

If there are four Numbers so stated, that the two middle Terms are : *l* with the one Extreme, and *hl* with the other, these four Numbers are : : *l*.

Exam. 2 : 5 : : 8 : 20, are : : *l*; 2, 5, 8, are : *l*, and 5, 8, 20, *hl*. *Universally*, if *a*, *b*, *c*, are : *l*, and *b*, *c*, *d*, are *hl*, then, *a*, *b*, *c*, *d*, are : : *l*.

$a,\ b,\ c,\ \frac{br}{2b-c}=\frac{bc}{a}$ | *Demon.* Suppose *a*, *b*, *c*, are : *l*, then to *b*, *c*, a third *hl* is $\frac{bc}{2b-c}$ (by *Probl.* 1st, *Cor.* 1st.) but to *a*, *b*, *c*, a 4th : : *l* is $\frac{bc}{a}$, and this is equal to the other; for *a*, *b*, *c*, being : *l*, therefore $a=2b-c$, hence

$\frac{bc}{2b-c}=\frac{bc}{a}$, i. e. a, b, c, being $:l$, the ſame Number d which is a 3d hl, to b, c, is a 4th $::l$, to a, b, c.

$2b\text{-}c : b :: c : \frac{bc}{2b-c}$

Or alſo thus, ſuppoſe any two Numbers, b, c, a 3d $:l$ to c, b, is $2b-c$, and a 3d hl to b, c, is $\frac{bc}{2b-c}$, and theſe 4 are manifeſtly $::l$; for by the common Rules a 4th $::l$, to $\overline{2b-c}, b, c$, is $\frac{bc}{2b-c}$.

The *Reverſe* of this Theorem is alſo true, *viz.* that if four Numbers are $::l$, and if the two Means with one of the Extremes are $:l$, or hl, they are contrarily hl or $:l$ with the other Extreme. The Demonſtration of which is contained in the former; for whether we ſuppoſe three Terms $:l$, $\overline{2b-c}$, b, c, or three Terms hl, b, c, $\frac{bc}{2b-c}$ the 4th $::l$ will be $\frac{bc}{2b-c}$, or $2b-c$, which is contrarily hl: or $:l$ with the other Extreme.

SCHOLIUMS.

1ſt. As either the Theorem, or its Reverſe, are demonſtrated independently of one another, ſo the one being ſuppoſed true, the other may be demonſtrated by its Means.

1°. Suppoſe the Theorem true, the Reverſe is demonſtrated thus, if a, b, c, d, are $::l$, and a, b, c, $:l$, then ſuppoſe a third hl to b, c, is n; by the Theorem $a : b :: c : n$; but $a : b :: c : d$, hence $n=d$, and b, c, d, are hl. The Demonſtration will proceed the ſame way by firſt ſuppoſing a, b, c. to be hl.

2°. Suppoſe the Reverſe is true, the Theorem is demonſtrated thus; let a, b, c, be $:l$, and b, c, d, hl; then to a, b, c, let a 4th $::l$ be n, by Suppoſition b, c, n, are hl, (becauſe a, b, c, are $:l$, and a, b, c, n, are $::l$.) But ſo alſo are b, c, d, conſequently, $n=d$, and $a : b :: c : d$. If a, b, c, are ſuppoſed hl, the Demonſtration proceeds the ſame way.

2d. We may find Examples of this Theorem in Integers, by taking any three Integers, which are $:l$, and to them find a 4th $::l$. If this is an Integer, you have what's ſought; but if it's fractional, multiply all by the Denominator. Thus if a, b, c, are three Numbers $:l$, the 4th $::l$ is $\frac{bc}{a}$, and if this is not an Integer, then multiply by the Denominator a, and the four Numbers ſought are $aa : ab : ac : bc$.

THEOREM VI.

If there are four Numbers ſo ſtated that the Extremes with one Mean are $:l$, and with the other hl, (*i. e.* if betwixt any two Numbers you put an Arithm. and alſo an Harm. Mean) the four will be $::l$.

Exam. $6 : 8 :: 9 : 12$ are $::l$, $6, 9, 12$, are $:l$, and $6, 8, 12$, hl. *Univerſally*, if a, c, d, are $:l$, a, b, d, hl : then are a, b, c, d, $::l$.

$a : \frac{a+b}{2} :: \frac{2ab}{a+b} : b$

Demon. Betwixt a and b, a Mean $:l$ is $\frac{a+b}{2}$, and a Mean hl is $\frac{2ab}{a+b}$, but the Products of the Extremes and Means are evidently equal, therefore the four are $::l$.

The Reverſe of this Theorem is alſo true, *viz.* if four Numbers are $::l$, and if the 2 Extremes with the one Mean are $:l$ or hl, they are contrarily hl or $:l$ with the other Mean;

Mean; the Demonstration of which is contained in the former, for a, $\frac{a+b}{2}$, b represent any three Numbers $::l$, and a, $\frac{2ab}{a+b}$, b any three hl; and if a, b, are the Extremes of four Numbers $::l$, and either of these other Expressions one of the Means, it's shown that the other of them will be the other Mean, which is an $:l$ or hl Mean betwixt the Extremes a, b, if the former is contrarily hl, or $:l$.

SCHOLIUM.

1. Suppose either the Theorem or its Reverse true, the other may be demonstrated by means of it, after the same manner as was done in the last Theorem.

2. We may find as many Examples of this Theorem as we please in Integers, after this manner; take any two Integers, and betwixt them find a mean hl, which being a mixt Number, reduce all the three by its Denominator to other three Numbers, which shall be all Integers, and still hl (by *Theor.* 1st) then if the half Sum of the Extremes is an Integer, it is the Arithm. Mean, but if it's not Integral, double the three Terms already found, and then the half Sum of the Extreams will be an Integer.

Exam. If I take 3 and 5, a mean hl is $\frac{30}{8}$, therefore multiplying all by 8, I have 24, 30, 40, hl; and a mean $:l$ betwixt 24 and 40 is 32; therefore 24, 30, 32, 40, is an Example of what's required. But if I take 2, 3, their harm. Mean is $\frac{12}{5}$, and I reduce them to this Series 10, 12, 15; but here I cannot have a mean $:l$ in Integers, therefore I double these Numbers, making 20, 24, 30, and the mean $:l$ is 25; and so 20, 24, 25, 30, is an Example of what's required.

Hence again, If it were proposed to find out two Numbers, the half of whose Sum is an Integer, and also whose Sum is an aliquot Part of double the Product, it's plain from the preceding Demonstration, that if we find any Example of this Theorem, the Extremes are Numbers such as are here required; for the Extremes being a, b, the two Means are $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$.

3. This Theorem and the preceding may coincide, *viz.* there may be four Numbers $::l$, whereof the two Means may be $:l$ with the one Extreme, and hl with the other; and also the two Extremes may be $:l$ with the one middle Term, and hl with the other, as in this Example $2:3::4:6$; but this Coincidence does not always happen; for either of the Parts may be found by it self without the other. So these 6, 8, 9, 12, is an Example of this Theorem, but not of the other; as these 3, 6, 9, 18, or these 3, 4, 6, 8, belong to the former Theorem, neither of which belongs to this.

COROLL. If betwixt two Numbers, A, E, are put three Means, an Arithmetical (B), Geometrical (C), and Harmonical Mean (D), these Means are in Geometrical Progression; the Geometrical Mean being the middle of the three; for $A\times E=B\times D$, also $A\times E=C\times C$, therefore $B\times D=C\times C$, or $B:C::C:D$.

THEOREM VII.

If four Numbers are so stated that the two Means with the one Extreme are hl, and with the other $::l$, these four are hl.

Exam. 3, 4, 6, are hl; 4, 6, 9, are $::l$, and 3, 4, 6, 9, are hl. *Universally*, if a, b, c, are $::l$, and b, c, d, hl, then a, b, c, d, are hl.

$a,\ b,\ c,\ \frac{bc}{2b-c}=\frac{ac}{2a-b}$

Demon. Since a, b, c, be $::l$, and to b, c, a third hl is by Supposition possible; this by *Problem* first is

bc

$\frac{bc}{2b-c}$; alſo to a, b, c a fourth hl is poſſible (*Sch. Coroll.* I. *Prob.* I.) and it is $\frac{ac}{2a-b}$; which I demonſtrate to be equal to $\frac{bc}{2b-c}$; *thus*; $a : b :: b : c$ by Suppoſition; hence $2a : 2b :: b : c$, and $2a-b : 2b-c :: b : c$. But $b : c :: bb : bc$; hence $2a-b : 2b-c :: bb : bc$; and again, $bc : 2b-c :: bb : 2a-b$, therefore $\frac{bc}{2b-c} = \frac{bb}{2a-b}$; but $bb = ac$, therefore $\frac{bc}{2b-c} = \frac{ac}{2a-b}$.

Or alſo thus: To any two Numbers, as b, c, a third hl is $\frac{bc}{2b-c}$, and a third $::l$ to c, b, is $\frac{bb}{c}$, and theſe four are hl, *viz.* $\frac{bb}{c}$, b, c, $\frac{bc}{2b-c}$; a fourth hl to $\frac{bb}{c}, b, c$, being $\frac{bc}{2b-c}$ by *Probl.* I. for $\frac{bb}{c} \times c = bb$, and $\frac{2bb}{c} - b = \frac{2bb-bc}{c}$, and $bb \div \frac{2bb-bc}{c} = \frac{bbc}{2bb-bc} = \frac{bc}{2b-c}$.

The Reverſe of this Theorem is alſo true, *viz.* if four Numbers are hl, and the two Means with one Extreme $:l$, or hl, it will be contrarily hl or $:l$ with the other Extreme.

The Demonſtration is contained in the former: For whether we ſuppoſe three Numbers $::l$, which may be expreſſed $\frac{bb}{c} : b : c$; or three Numbers hl which may be expreſſed $b, c\ \frac{bc}{2b-c}$; the fourth hl will be $\frac{bc}{2b-c}$ or $\frac{bb}{c}$, contrarily hl or $:l$, with the other Extreme.

Schol. Either the Theorem or the Reverſe being ſuppoſed true, the other may be demonſtrated by means of it in the Manner ſhown in *Theorem* V.

PROBLEM III.

To find a fourth Contra hl *to three given Numbers.*

Caſe 1ſt. If the firſt Term is leſs than the ſecond, the Rule is this; from the Product of the firſt and ſecond Terms ſubſtract the Square of the firſt Term; and to the Difference add the fourth Part of the Square of the third Term; out of this Sum extract the ſquare Root, to which add the half of the third Term; this laſt Sum is the fourth Term ſought.

Exam. To theſe d, c, b (d being leſs than c) ſuppoſe a fourth Contra hl is a, then is $a = \frac{b}{2} + \overline{dc - dd + \frac{bb}{4}}\big|^{\frac{1}{2}}$.

Exam. in Numbers. To theſe three Numbers, 2, 8, 4, a 4th Contra hl is 6; for $2 : 6 :: 6-4 : 8-2$; which I find thus; $2 \times 8 = 16$, and $2 \times 2 = 4$, then $16 - 4 = 12$; again, $4 \times 4 = 16$, whoſe fourth Part is 4; which added to 12, the Sum is 16, whoſe ſquare Root is 4, to which add 2 (the half of 4) the Sum is 6.

Caſe 2d. If the firſt Term is greater than the ſecond Term, the Rule is this: From the Square of the firſt Term ſubſtract the Product of the firſt and ſecond; and this Difference ſubſtract from the fourth Part of the Square of the third Term; then extract the

Square Root of this last Difference; which add to or substract from the Half of the third Term; the Sum or the Difference (one of the two) will be the fourth sought: And you must chuse that which makes the Proportion with the given Numbers.

Exam. To these a, b, c, (a being greater than b) a fourth Contra hl is $\frac{c}{2}+\overline{\frac{cc}{4}-aa-ab}\big|^{\frac{1}{2}}$; or $\frac{c}{2}-\overline{\frac{cc}{4}-aa-ab}\big|^{\frac{1}{2}}$.

Example in Numbers. To these 6, 4, 8, a fourth Contra hl is 2, thus found; $6\times4=24$, and $6\times6=36$, then $36-24=12$; again, $\frac{8\times8}{4}=\frac{64}{4}=16$, and $16-12=4$, whose Square Root is 2, then $\frac{8}{2}=4$; Lastly, $4+2=6$, and $4-2=2$, which last is the Number sought.

Demon. Let these be Contra hl, a, b, c, d; and suppose a greater than b, and c than d; then it is $a:d::c-d:a-b$; and multiplying the Extremes and Means, it is $aa-ab=dc-dd$. But it has been demonstrated (*Probl.* VI. *Book* III.) that if $aa-ab=p$, then is $a=\frac{b}{2}+\overline{p+\frac{bb}{4}}\big|^{\frac{1}{2}}$, which is the Rule of *Case* 1st, supposing $dc-dd=p$. Again, it is also demonstrated, that if $dc-dd=p$, then is $d=\frac{c}{2}\pm\overline{\frac{cc}{4}-p}\big|^{\frac{1}{2}}$, which is the Rule of *Case* 2d, supposing $aa-ab=p$. It is demonstrated also that if $\frac{cc}{4}$ be greater than p, so will $\frac{c}{2}$ be greater than $\overline{\frac{cc}{4}-p}\big|^{\frac{1}{2}}$. But if p is not less than $\frac{cc}{4}$, the *Problem* is impossible. Also in both Cases, if the Square Root to be extracted is surd, there is no fourth Contra hl in rational Numbers.

Coroll. If to two given Numbers a third Contra hl is required, the preceding Rules are applicable, by supposing the second and third Terms to be the same. Thus; to these a, b, a third Contra hl, when a is less then b, is $\frac{b}{2}+\overline{ab-a^2+\frac{b^2}{4}}\big|^{\frac{1}{2}}$; and if a is greater than b, then it is $\frac{b}{2}\pm\overline{\frac{bb}{4}-aa-ab}\big|^{\frac{1}{2}}$, as you will find in this Example; 3, 5, 6, where $3:6::6-5:5-3$, or $3:6::1:2$.

PROBLEM IV.

To find a Contra hl *Mean betwixt two given Numbers.*

Rule. Divide the Sum of their Squares by their Sum; the Quote is the Mean sought.

Exam. Betwixt a, b, a Contra hl Mean is $\frac{aa+bb}{a+b}$.

Example in Numbers. Betwixt 3, 6, a Contra hl Mean is $5=\frac{9+36}{9}=\frac{45}{9}$.

Demon. If these three are Contra *hl*, a, b, c, that is, if $a : c :: b-c . a-b$, then multiplying the Extremes and Means, it is $a^2-ab=bc-c^2$; and adding c^2 to both Sides, it is $bc=a^2-ab+c^2$; and again, adding ab to both Sides, it is $ab+bc=a^2+c^2$; and dividing both Sides by $a+c$, it is, $b=\frac{a^2+c^2}{a+c}$.

THEOREM VIII.

If four Numbers are so stated that the Extremes with one Mean are *hl*, and with the other Contra *hl*, *i. e.* if betwixt two Numbers you put a Mean *hl*, and another Contra *hl*, these four Numbers are : *l*.

Exam. 3, 4, 6, are *hl*, and 3, 5, 6, Contra *hl*, and 3, 4, 5, 6, : *l*. *Universally*; if a, b, c, d are *hl*, and a, c, d Contra *hl*, then are a, b, c, d, : *l*.

$$a . \frac{2ab}{a+b} : \frac{a^2+b^2}{a+b} . b$$

Demon. Betwixt a, b, a Mean *hl* is $\frac{2ab}{a+b}$ (*Probl.* II.) and a Mean Contra *hl* is $\frac{a^2+b^2}{a+b}$ (*Probl.* IV.) and these four are : *l*; viz. a; $\frac{2ab}{a+b}$; $\frac{a^2+b^2}{a+b}$; b: for $a-\frac{2ab}{a+b}=\frac{a^2+b^2}{a+b}-b=\frac{a^2-ab}{a+b}$, by the common Rules: Or thus, $a+b$ (the Sum of the Extremes) is $=a^2+b^2+2ab$, ($=a+b$ squar'd) divided by $a+b$, the common Denominator; which Quote is the Sum of the middle Terms.

The *Reverse* of this *Theorem* is also true, *viz.* That if four Numbers are : *l*, and the Extremes with one Mean are *hl*, or Contra *hl*; with the other Mean they will be contrarily Contra *hl*, or *hl*. The Demonstration is contained in the former: Thus; any three Numbers *hl* may be represented $a : \frac{2ab}{a+b} : b$, and any Contra *hl*, $a : \frac{a^2+b^2}{a+b} : b$; and if a, b are the Extremes of four Numbers : *l*, where one of the Means is $\frac{2ab}{a+b}$, or $\frac{a^2+b^2}{a+b}$, an *hl*, or Contra *hl* Mean betwixt the Extremes; it's demonstrated that the other of the two Means will be the other of these two Expressions, *viz.* contrarily an *hl* or : *l* Mean betwixt the same Extremes: Or thus; let a, b, c, d, be : *l*, and a, b, d, be *hl*, i. e. $a : d :: a-b : b-d$; then because $a-b=c-d$, also $a-c=b-d$, therefore $a : d :: c-d : a-c$, i. e. a, c, d, are Contra *hl*; or if a, b, c, are ÷*l*, the Demonstration will proceed the same Way.

COROLLARIES.

1. Hence we have another Method for finding a Mean Contra *hl*: Thus, find an Arithm. Mean; then from the Sum of the Extremes take this Mean; the Remainder is the Mean Contra *hl*: Because the four being : *l*, the Sum of the Extremes and Means are equal.

2. The *hl*, : *l*, and Contra *hl* Means, betwixt two Numbers, are in Arithm. Progression; for if betwixt A, E, the Mean *hl* is B, the : *l*, C, and Contra *hl*, D; then because A, B, D, E and also A, C, E are : *l*; hence A+E=B+D, and A+E=2C; therefore B+D=2C, or B, C, D, are ÷*l*.

THEOREM IX.

Of the mean Proportionals betwixt two Numbers already explained, the Order is this: The direct Harmonical is the least Number; then follow in Order the Geometrical, the Arithmetical, and the Contra *hl*.

2	:	$3\frac{1}{5}$	:	4	:	5	:	$6\frac{2}{3}$	:	8
10	·	16	·	20	·	25	·	32	·	40
A	:	B	:	C	:	D	:	E	:	F

Or thus,

$$a : \frac{2ab}{a+b} : \overline{ab}|^{\frac{1}{2}} : \frac{a+b}{2} : \frac{a^2+b^2}{a+b} : b$$

hl.	*Geom*	*Arith.*	*Con. hl.*

Demon. 1°. It's already shewn in *Theo.* XXI. Chap. V. that the Arithmetical Mean D is greater than the Geometrical C; and by *Coroll.* to *Theorem* VI. preceding, it's shewn, that B, C, D are $\div$ *l*, therefore D and C are both greater than B, *i. e.* the Harmonical Mean is less than either the Geometrical or Arithmetical, and so these three Means are in the Order here stated.

2°. The Contra *hl* Mean is greater than the Arithmetical; for by *Coroll.* 2. to the last *Theorem*, the *hl*, : *l*, and Contra *hl* Means are $\div$ *l*, the Arithmetical being the Middle of the three; and by the Expressions of these Means it will be easily shewn, that the Contra *hl* Mean $\frac{a^2+b^2}{a+b}$ is greater than the *hl*; $\frac{2ab}{a+b}$; from which it is plain, that the Contra *hl* is greater than the Arithmetical. The former is proved thus, the Denominators are equal, therefore we have only the Numerators $2ab$, and a^2+b^2 to compare; now $a^2 : ab :: ab : b^2$ (both being as a, b) and $a^2+b^2 \urcorner 2ab$ (*Theo.* XXI. Chap. V.) therefore the Contra *hl* Mean is greater than the *hl*, and consequently than the Arithmetical; so that the four Means are in the Order proposed.

General Scholium I.

Comparing the three Kinds of Proportions, Arith. Geom. and harm. we have this very remarkable Difference to observe, *viz.* That a Progression Geom. can be continued from a given Number, upwards or downwards, in Infinitum; an Arithmetical Upwards but not Downwards; a direct harmonical Downwards but not Upwards; and a Contra *hl* neither Ways.

II.

As the Harmonical Proportions of three Numbers already explained, proceed from a Comparison of the Ratio of the Extremes with the Ratio's of the Differences betwixt the Extremes and the middle Term; so there are other Comparisons that may be made in the same general Way; that is, by comparing the Ratio of any two of three Numbers, with the Ratios betwixt the Differences of any one of the three, and the other two. But as I design no particular Consideration of these (since after what is done of this Kind, any may do what more they please) I shall only lay down Examples of the Comparisons wherein it is possible there may be a Proportionality, and shew where it cannot be.

Let a, b, c, be three Numbers, whereof a is the least, and c the greatest; then is it possible to invert them so as the following Numbers be :: *l*, *viz.*

	Examples	$a \cdot b \cdot c$
$a : c :: b-a : c-b$		$3 \cdot 4 \cdot 6$
$a : c :: c-b : b-a$		$3 \cdot 5 \cdot 6$
$a : c :: b-a : c-a$		$6 \cdot 8 \cdot 9$
$a : c :: c-b : c-a$		$6 \cdot 7 \cdot 9$

	Examples	$a \cdot b \cdot c$
$a : b :: b-a : c-b$		$4 \cdot 6 \cdot 9$
$a : b :: c-b : b-a$		$6 \cdot 9 \cdot 11$
$a : b :: b-a : c-a$		$4 \cdot 6 \cdot 7$
$a : b :: c-b : c-a$		$3 \cdot 4 \cdot 7$

	Examples	
$b : c :: b-a : c-b$		$4 : 6 : 9$
$b : c :: c-b : b-a$		$1 : 4 : 6$
$b : c :: c-b : c-a$		$8 : 9 : 12$
$b : c :: b-a : c-a$		*Impossible.*

Upon

Upon these Examples *observe*, That they are distinguished into three Cases, according as $a : c$ or $a : b$, or $b : c$, are made the first and second Terms: Then in every Class there are four Cases; and in the two first of these, the third and fourth Terms are the same, only reversed; but of the third and fourth Cases the fourth Terms only are common; nor can a Proportion be stated with these reversed, while the first and second Terms keep the same Order, for then the Comparison would be dissimilar. Thus it is impossible, that it should be $a : c :: c—a : c—b$, because a is less than c; but $c—a$ is greater than $c—b$, since b is greater than a. Then the last Case of the third Class is marked impossible; for if the same Number a is taken from each Term of any Ratio of Inequality, the Remainders cannot be in the same Ratio; since the Remainders cannot be in the same Ratio unless the Numbers taken away be in the Ratio of the whole, (*Coroll.* II. *Theo.* V. Ch. III.)

Observe also, That in the first Case of the second and third Classes, the Examples are always $\div l$, as will be manifest from the Consideration of *Theorem* V. Chap. 3. and so any three Numbers $\div l$ are Examples of these Cases.

III.

Of harmonical Progressions there is another Kind than that already explained, wherein every three adjacent Terms are directly *hl*; for a Series of direct Harmonicals may be found such that every four adjacent Terms are *hl*, as in this Example, $6 : 8 : 10 : 15 : 20 : 40$.

But then *observe* this great Difference betwixt the Nature of these harmonical Progressions, and the Geometrical and Arithmetical; *viz.* that in these last, because every three Adjacent are Proportional, therefore so are every four either Adjacent or taken two and two at equal Distance; but it's not so in the harmonical Kinds, for tho' every three Adjacent are *hl*, yet every four will not be so, neither will the Conclusion hold from four adjacent Terms to three.

§. 2. *Of the Name and Application of* Harmonical Proportion.

I HAVE already said, that the Name comes from the Application of this kind of Proportion found in *Musick*: Not as if this were the only Proportion found among musical Sounds; but because its Effects are the most perfect. It would be out of my Road to say much on that Subject here; for I might as well pretend to explain all the Subjects to which the other Proportions are applicable: Yet this being a Thing little considered (though the Writers on Musick have fully explained it) and of great Use and Curiosity; I shall say as much upon it as may serve to give a distinct Idea of this Application, to such, at least, who have made the following Observations, or can distinctly conceive them.

1*st*. The Thing in Sounds, upon which what we call Harmony in Musick depends, is that Property of them whereby they are distinguished into High and Low; called also Acute and Grave; the Idea of which we get by a Series of Notes or Sounds raised one after another upon a musical Instrument, or by a Voice. And observe also, That this Highness or Acuteness is very different from the Strength or Loudness of a Sound; for the Voice of a Boy may be Acuter, though not so Strong or Loud as that of a Man.

2*d*. Take two Strings (fit for a musical Instrument) which are equal, or the same in all respects, except the Lengths [*i.e.* of the same Matter, and Dimensions, and equally stretched: And this will necessarily happen, if we take a String stretched to any Degree fit for sounding: Then divide it into any two unequal Parts, which may be sounded separately; which is done by having the String fix'd at both Ends, and a little raised over

over the Surface of any Instrument or Table; and setting under it, in any Point that divides it unequally, a Bridge, so that the String is not the more stretched by it, but the two Parts so separated that they can be sounded each by it self.] The longer String will give the Lower (or Graver) Sound; and the shorter, the Higher (or Acuter) Sound; so that a Number of Strings of different Lengths (all other Circumstances being the same) will give a Series of different Sounds or Notes, Rising or Falling, in Acuteness and Gravity, as the Strings become shorter or longer.

3*d*. A String may be made of such a Length, and so stretched, as that its Sound shall have the same Degree of Acuteness (or Gravity; it's no Matter which we say, since they are only Words expressing a Relation of one Sound to another) with any other Sound; and consequently, any two Sounds may be expressed by two Strings, the same in all Respects but the Lengths; and then the Relation of their Lengths may very fitly be considered as expressing the Relation of these two Sounds, as to Acuteness or Gravity, which we also call the Relation of their Tones; for every Relation must have some real absolute Foundation; and this in the Relation of Acuteness and Gravity among Sounds we call the Tone. For though every Sound is both Acute and Grave in respect of different Sounds; yet every Sound must have its own determinate Degree and Measure of that upon which Acute or Grave depend; which are but relative Names for different Degrees of it compared to one another. What this Tone depends upon more immediately we shall next consider.

4*th*. Though the different Lengths of Strings (other Circumstances being the same) produce Sounds of different Tones; or Acute and Grave in respect to one another; yet the Relations and Degrees of Tone are not measured by the simple Differences of the Lengths of the Strings; so that though several Strings have an equal Difference of Lengths, as if they were 8, 6, 4, 2; yet their Sounds do not exceed each other by equal differences of Tone; but the Relation of the Tone is the Geometrical Relation of the Lengths of the Strings; so that to make a Series of Sounds rise or become Acuter by equal Differences, the Lengths of the Strings must be in Geometrical Progression, as 8 : 4 : 2 : 1. Now for the Reason of this, observe, that when a String is sounded it is put into a Motion, which we call Vibratory, *i. e.* to and again; and the Vibrations, or Motions to and again, are quicker or slower as the String is shorter or longer (other Circumstances being alike.) And the Mathematicians have demonstrated that the Number of Vibrations of two such unequal Strings made in the same Time, are in the Ratio of their Lengths reciprocally; thus, one String being one Foot long, and another two, the former makes two Vibrations in the Time that the other makes one. And since all Sound is produced by the vibratory Motion of the Parts of Bodies, they conclude, that the Tone of every Sound depends immediately upon the Number of Vibrations made in any Time; and since we can express the Tone of any Sound by a String, and consequently of any Number of different Sounds, by as many Strings differing only in Length; the Vibrations of these Strings are the same as the Vibrations of the Parts of other Bodies, whose Tones they are equal to; and because the Vibrations are reciprocally as the Lengths, therefore the Ratios of the Lengths of Strings (*cæteris paribus*) are true Expressions of the Ratios of Tone; so that where-ever betwixt any two such Strings, there is the same Ratio of Lengths, there must be the same Ratio of Tone, *i.e.* the same Excess of the one above the other: For *Example*, If four Strings are 8 : 4 : : 6 : 3, as far as the Tone of 4 is above that of 8, so far is the Tone of 3 above that of 6; because they are in the same Ratio 2 : 1. And so in these, 8 : 4 : 2; where if we judged by the simple Differences of the Numbers, the Tone of 2 would exceed that of 4, only by the Half of what the Tone of 4 exceeds that of 8; whereas the Excess of Tones is equal, the Ratios being so.

In

In the preceding Obſervations you have the general Grounds of the Arithmetical Theory of Muſick; or the Foundation upon which muſical Sounds fall under Arithmetical Calculation: The following ſhall finiſh what I have to ſay upon this Subject, and ſhew you the Application of Harmonical Proportion.

5th. Sounds differing in Tone are applied in Muſick two Ways, *viz.* In Succeſſion and Conſonance; *That is*, by raiſing diſtinct Notes one after another; and by mixing or joining them together ſo that they fall upon the Ear all at once. But then *obſerve*, That every Difference of Tone is not fit for Muſick; or, any Notes taken in any Relations of Tone, and in any Order, cannot pleaſe the Ear, either in Succeſſion or Conſonance; but there are certain Relations upon which Muſick depends, and without which it has no Being; and theſe Experience has diſcovered and approved; *that is*, it is found that Sounds in certain Relations of Tone, being heard together, or one after another, have ſuch an Agreement or Union as to pleaſe the Ear, but in different Degrees, according to the Relations. Every Relation that produces an agreeable Conſonance will alſo make an agreeable Succeſſion, but not always the contrary; and therefore the former are reckoned the fundamental and eſſential Principles of Muſick; and ſuch Sounds are particularly called *Concords*; the contrary Effect being called Diſſonance or Diſcord. Though there be different Degrees of Concord, according as the Relations differ, yet to our preſent purpoſe it's enough to take Notice of what Muſicians call the Simple, Primitive, or Original Concords, of which there are only ſeven; expreſſed by the Ratios (of Numbers) and Names in this Table; to be underſtood thus,

2 : 1 *Octave*,	If two Strings differ only in Length, then their Lengths
3 : 2 *Fifth*,	being in any of theſe Ratio's; for *Example*, as 2 : 1 or 3 : 2,
4 : 3 *Fourth*,	*&c.* their Tones are Concord, and the Agreeableneſs is ac-
5 : 4 *Third greater*,	cording to the Order here expreſſed; the *Octave*, 2 : 1 being
6 : 5 *Third leſſer*,	the moſt perfect, then the *Fifth* 3 : 2, and ſo on (tho' there
5 : 3 *Sixth greater*,	is ſome Queſtion, Whether the *Third leſſer*, or *Sixth greater*,
8 : 5 *Sixth leſſer*.	is preferable.) Or if we would expreſs all theſe Concords in

relation to one fundamental Sound, which we may expreſs by 1, the Series of Sounds having theſe gradual Concord-Relations to that Fundmental, is expreſſed thus,

$$1 : \frac{1}{2} : \frac{2}{3} : \frac{3}{4} : \frac{4}{5} : \frac{5}{6} : \frac{3}{5} : \frac{5}{8}$$

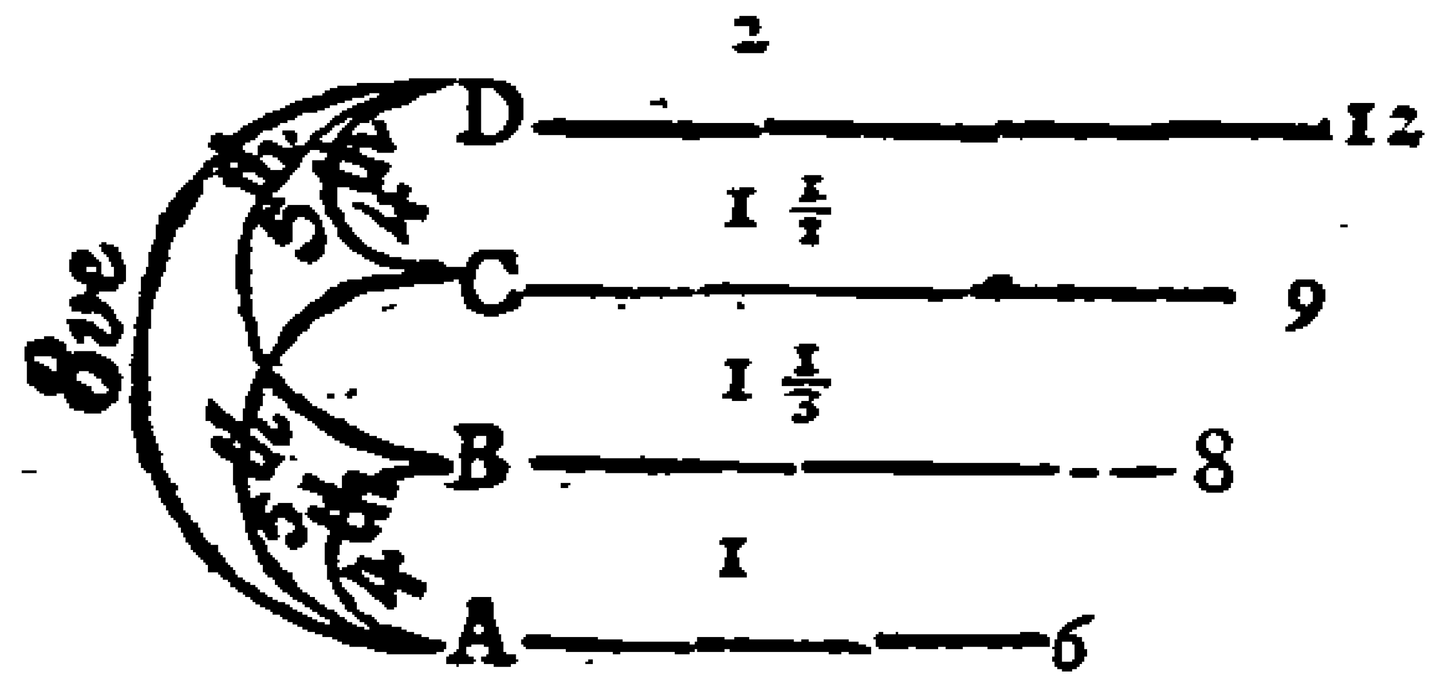
Fundam, *Octave*, *Fifth*, *Fourth*, *Third greater*, *Third leſſer*, *Sixth greater*, *Sixth leſſer*.

When two Sounds have the ſame Tone, they are ſaid to be Uniſons; which certainly is the firſt and moſt perfect degree of Concord; yet more commonly Concord is applied to Sounds of different Tone. The Reaſon of the Names *Octave*, &c. you'll find below.

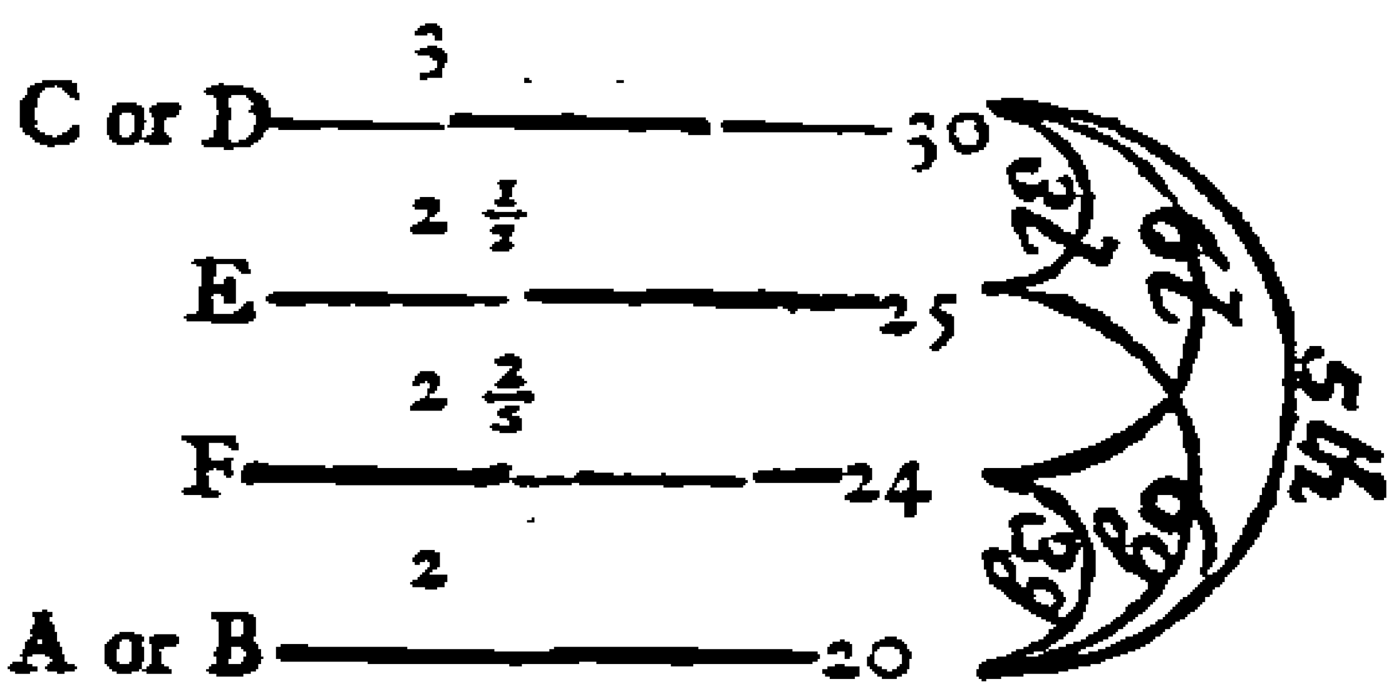
Now as theſe Concord-Relations are the fundamental and eſſential Principles of Muſick; ſo the Thing remarkable to our Purpoſe here is, their Connection and Dependance upon one another; in which the Application of the Proportions Arithmetical, Geometrical and Harmonical is to be found. *Thus*,

Let

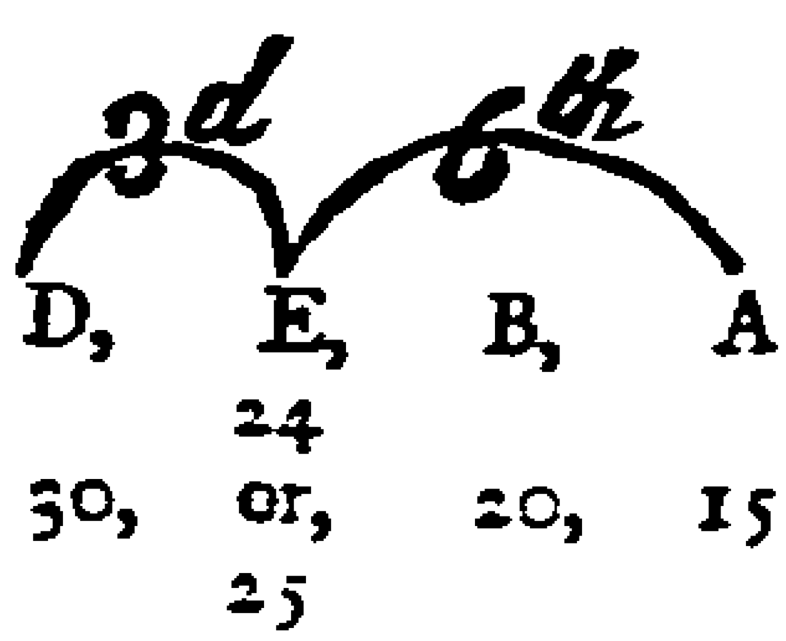
Let any two Strings, D and A be in length, as 2 : 1 (*cæteris paribus*) they make the Concord *Octave*, as above. Betwixt 2, 1, take an Arithmetical Mean C, which is $1\frac{1}{2}$; this String will make with D the Concord, Fourth, (for 2 : $1\frac{1}{2}$ :: 4 : 3) and with A the Concord, Fifth, (for $1\frac{1}{2}$: 1 :: 3 : 2.) Again, take an Harmonical Mean betwixt D and A, it is B = $1\frac{1}{3}$, which makes with D a Fifth (for 2 : $1\frac{1}{3}$ :: 3 : 2) and with A a Fourth, (for $1\frac{1}{3}$: 1 :: 4 : 3.) And the Lengths of theſe four Strings being reduced to Integral Expreſſions, are 12, 9, 8, 6; which are in Geometrical Proportion; for 12 : 9 :: 8 : 6, or 12 : 8 :: 9 : 6, in which 12 : 9 and 8 : 6 are both Fourths; and 12 : 8, 9 : 6 both Fifths.



Again, take the Fifth 3 : 2; an Arithmetical Mean, E=$2\frac{1}{2}$, makes the Third leſſer with D, and Third greater with B. But take an Harmonical Mean, F=$2\frac{2}{5}$, it makes a Third greater with D, and a Third leſſer with B. And being reduced to Integers, they are 30, 25, 24, 20, which are in Geometrical Proportion; for 30 : 25 :: 24 : 20, or 30 : 24 :: 25 : 20; in which 30 : 25 and 24 : 20 are both Thirds leſſer; and 30 : 24, 25 : 20 both Thirds greater.



Now, as the two Concords next in Perfection to the *Octave* ariſe immediately from the Diviſion of that Concord, *i. e.* putting an Arithmetical or Harmonical Mean betwixt the Extremes of the *Octave*; ſo the Fifth being the ſame way divided, produces the two next Concords of Third greater and leſſer. Then for the Sixths greater and leſſer; they are the Conſequences of the preceding Diviſions; for having divided the *Octave* into the Fifth and Fourth; then the Fifth into the two Thirds; we have alſo the Sixths: *Thus*. If D, A are *Octave*, and D, B a Fifth, alſo D, E a Third greater or leſſer; then E, A will be contrarily a Sixth leſſer or greater, as the Numbers annexed do ſhew; for if E=24, then, as 30 : 24 :: 5 : 4 a Third greater; ſo 24 : 15 :: 8 : 5 a Sixth leſſer. And if E=25, then as 30 : 25 :: 6 : 5 a Third leſſer; ſo 25 : 15 :: 5 : 3 a Sixth greater.



I ſhall go one ſhort Step further, and ſhew the Reaſon of theſe Names, *Octave*, *Fifth*, &c. which will ſhew a further Application of the Harmonical Proportion. Let A : H be an *Octave*; A : D a Fourth, and A : E a Fifth; then A : C a Third greater; alſo D : F a Third greater, (from the Diviſion of the Fifth D : H) then will F : H be a Third leſſer, and conſequently A : F a Sixth greater.

A	B	C	D	E	F	G	H
1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{15}$	$\frac{1}{2}$

Again, A : C being a Third greater, as 5 : 4, take B a Harmonical Mean, then will A to B be as 9 : 8, and B to C as 10 : 9. *Laſtly*, Let G be taken in the ſame Ratio to F, as B to A, and its Length will be $\frac{8}{15}$ for 9 : 8 :: $\frac{3}{5}$: $\frac{8}{15}$. And then we have eight Sounds in ſuch Relations of Tone to one another, as make the Series or Succeſſion of Sounds, which is called the *natural Scale* of Muſick, which contains in it all the Principles of *Harmony*; wherein, beſides the Concord-Relations, theſe are alſo very con-

considerable, which are betwixt the several intermediate Sounds, as A · B, B · C, *&c.* which are called the Degrees of the *Scale*; of which there are but three different ones, *viz.* that of A to B, as 9 : 8; of B to C, as 10 : 9; of C to D, as 16 : 15. That the rest are the same, and in what Order they are, you see by comparing their Expressions: Or see them here, where the Lengths of the Strings are set above them, and their mutual Relations betwixt them below. So B is $\frac{8}{9}$ of A, C is $\frac{9}{10}$ of B, D is $\frac{15}{16}$ of C, and so on.

$$\begin{array}{ccccccccccccccc}
1 & & \frac{8}{9} & & \frac{4}{5} & & \frac{3}{5} & & \frac{2}{3} & & \frac{3}{5} & & \frac{8}{15} & & \frac{1}{2} \\
A & \cdot & B & \cdot & C & \cdot & D & \cdot & E & \cdot & F & \cdot & G & \cdot & H \\
 & \frac{8}{9} & & \frac{9}{10} & & \frac{15}{16} & & \frac{8}{9} & & \frac{9}{10} & & \frac{8}{9} & & \frac{15}{16} &
\end{array}$$

Now the Relations of each of these Sounds to the fundamental A being named according to the Number of Notes from A in the Scale, hence are the Names of Third, Fourth, *&c.* So the Relation 5 : 4 is called a Third, because it's that betwixt A : C, which having one Note betwixt them make in all three Notes of the Scale; for the same Reason 6 : 5 is called also a Third, being the Ratio betwixt C and E. But 5 : 4 is a greater Ratio, and therefore called the Third greater, as the other 6 : 5 is called the Third lesser. For the like Reason, 4 : 3, which is betwixt A : D, is called a Fourth; 3 : 2, which is betwixt A : E, is called a Fifth. 3 : 5 betwixt A : F is called a Sixth greater; 8 : 5 betwixt C : H is called a Sixth lesser; and 2 : 1 betwixt A : H is called an *Octave.* Then for the intermediate Degrees, they are called Seconds, whereof 9 : 8 is the greatest, 10 : 9 next, and 16 : 15 least. And they are otherwise called, particularly, 9 : 8 a greater Tone, 10 : 9 a lesser Tone, 16 : 15 a Semi-Tone, (the Word Tone being here taken in another Sense than we formerly used it in.) And the Ratio 15 : 8 betwixt A : G is called a Seventh. But I shall insist no more upon this Subject, having done all that was proposed, or could be expected in this Place.

APPENDIX

TO

BOOK IV.

Containing ſome further CONSIDERATIONS concerning the Doctrine of *Ratios* and *Proportion* (Geometrical.)

§. 1. *Of Quantities Commenſurable and Incommenſurable, and their* Ratios : *Shewing how the whole Doctrine of* Ratios *and* Proportion *is reduced to the Science of Numbers.*

THE two grand Branches of *Pure Mathematicks* are *Arithmetick* and *Geometry*. As the Object of the firſt is *Number*, called alſo *Quantity* diſcontinued ; ſo the Object of the other is called *Magnitude*, or alſo *Quantity* continued ; the Species of which are *Lines*, *Surfaces* and *Solids*. Theſe two Branches are ſo connected, that the firſt is neceſſarily ſubſervient to the other, which without it would be uſeleſs, or rather could have no Being at all ; for the Conſideration of Numbers runs through the whole Science of Geometry. Among the Principles that are common to both Subjects, the moſt remarkable are contained in the Doctrine of Ratios and Proportion; whoſe Truths hold equally in Arithmetical and Geometrical Quantities : So that this is an univerſal Doctrine in Mathematicks, equally applicable to both its Branches ; but the Way of explaining it, ſo as it may comprehend both, has been Matter of Controverſy among the Mathematicians ; while ſome have approved *Euclid*'s Method (in the fifth Book of his Elements) and others have cenſured it ; ſome of them objecting againſt the very Foundations of his Method, and others complaining only of the Tedіouſneſs and Difficulty of it. To my preſent Purpoſe ; I have only theſe two Things to obſerve :

1ſt. That as I have never been able to help thinking *Euclid*'s Method tedious and difficult (without Regard to other Objections ; for I enter no further into the Controverſy : Thoſe who are curious may ſee Dr. *Barrow*'s learned Defence of *Euclid*, in his Mathematical Lectures) ſo I readily own they have been juſtly blamed, who for the univerſal Doctrine of Proportion, have only given us that of Numbers ; without explain-

explaining how the same is applicable to all Kinds of Geometrical Quantities. And,

2*d.* As I think this may be done in a very reasonable Sense; so it will make much easier Work than *Euclid*'s Method; and reduce the whole Doctrine of Ratios and Proportion to the *Science* of Arithmetick; by bringing the Relations of all Quantities under the Notion of Numeral Relations, in such a Sense that the same Truths may be applied in the same Demonstration to all Kinds of Quantities; whereby the Method is truly universal.

In order to this I shall first consider the Nature of Geometrical Quantities, as they are distinguished into *Commensurable* and *Incommensurable*: Thus,

One *Magnitude (Line, Surface,* or *Solid)* may be equal to a certain Part or Parts of another (of the same Kind;) and hence 'tis plain that the Definition of Geometrical Relation, applied to Numbers, is equally applicable to such two Magnitudes, and resolves into the same, as the Relation of one Number to another; so if the one is $\frac{a}{b}$ Parts of the other, they are to one another as a to b; for the one being divided into as many Parts as a expresses, the other is divisible into a Number of the same Parts equal to b: And these two Magnitudes are hence said to be *Commensurable*; because the same Magnitude is an *aliquot* Part to both, and therefore is contained in each of them a certain Number of Times without a Remainder; from which it is said to measure them both, and they to be Commensurable. Thus in the present Case, the same Magnitude is $\frac{1}{a}$ part of the one, and $\frac{1}{b}$ part of the other.

But every two Geometrical Quantities (of the same Kind) are not Commensurable, or have not a common *aliquot* Part; (as the *Geometers* have found and demonstrated in many Cases). Therefore the Definition of Geometrical Relation that agrees to Numbers and Commensurable Quantities, can not be accurately and strictly apply'd to Incommensurable Quantities: For though there is a Relation of containing and being contained betwixt two such Magnitudes, yet it is not such that we can say the lesser is precisely such a Part or Parts of the other; and therefore is not the same precisely as that of one Number to another; for if it were so, the Magnitude represented by the Unit composing these Numbers, would be an *aliquot* Part of both; and so they were not Incommensurable.

But though Incommensurable Quantities are not accurately as Number to Number, yet they come infinitely near to that Condition. To understand this, I shall first observe, That the simple Idea of being Incommensurable, or having no common *aliquot* Part, though it sufficiently distinguishes them from Commensurable Quantities, yet does not of it self give us any such Idea of their Relation or Manner of containing one another, as to distinguish the Relations of several Incommensurables; because they are all equal and alike in this Respect: Yet from this general Notion of Incommensurability we have a plain Consequence which furnishes such an Idea of their Relations, as sufficiently distinguishes them, and contains a Character or Mark of their Equality and Inequality, upon which to ground an Idea of Proportionality; which Consequence is this:

If two Quantities A, B, have not a common *aliquot* Part; then this necessarily follows, that the lesser, A, being taken out of the greater, B, as oft as possible, there is a Remainder; which Remainder being taken out of A, as oft as possible, there is also a Remainder; and this Remainder being taken out of the last Remainder, as oft as possible, there is yet a Remainder: And going on in the same Manner, taking the last Remainder out of the preceding, there will still be a Remainder for ever, which grows less and less *ad Infinitum*: For if ever we come to a Division that leaves no

Remainder, then the Divisor must be a common Measure to the given Quantities, A, B; the Reason of which you see plainly in the Demonstration of the Rule for finding the greatest common Measure of two Numbers (in *Probl.* V. *Chap.* II. *Book* II.) which is equally applicable to any two Quantities.

Now here is one way of conceiving the Relation betwixt these two Magnitudes, *viz.* By the infinite Series of Quotes arising from these alternate Divisions; and therefore, if any other two Incommensurables, as C, D, being the same Way divided, give the same Series of Quotes *ad Infinitum*, their Relation is like and equal to the former; and consequently we may say these four Magnitudes are Proportional, *viz.* A : B : : C : D. But now, though we can thus in general conceive of the Equality and Inequality of the Relations of different Incommensurables (and have reduced them, in one Sense, to Arithmetick;) yet still we want some more particular Exponent of these Relations; by Means of which we may bring them more directly under the Notion of the Relations of Number to Number; so that the whole Doctrine of Proportion may be comprehended in the same Principles and Method of Demonstration, already used for Numbers and commensurable Quantities: And this may be done different Ways in Consequence of the preceding Principles.

1°. In the alternate Division of B by A, and A by the Remainder, and so on; the farther the Operation is supposed to be carried on, the Remainder becomes the lesser *ad Infinitum*; and consequently we come the nearer and nearer for ever to a Quantity which is a common Measure or *aliquot* Part to both A and B; for if we should come at last to a Division without a Remainder, the Divisor would be a common Measure to A, B; and consequently the lesser the Remainder is, the nearer is the Divisor to such a common Measure; and as the Remainders diminish infinitely, it follows that they approach infinitely to the Condition of a common Measure of A, B: Wherefore supposing A, B, both divided by these Remainders successively one after another, the Quotes will approach nearer and nearer *ad Infinitum* to true and compleat Expressions of these Magnitudes; for as the last Remainder may be supposed less than any assignable Quantity, so will the Quotes of A, B, by the last Divisor, express two Quantities that shall want less than any assignable Quantity of A and B; and consequently A and B are infinitely near in the Relation of these two Quotes, *i. e.* infinitely near, as Number to Number. Or we may conceive it also thus: Suppose any of the two Quantities, as B, divided into any Number of equal Parts, each of which is less than the other, A; then will A contain a certain Number of the same Parts, but with some Remainder over; else A, B were Commensurable: And the more Parts B is divided into, as that Part is smaller, so A contains the more of them, with a lesser Remainder: And by supposing the Number of Parts of B increased gradually *ad Infinitum*, the Number of the same Parts contained in A, does also increase, and the Remainder decreases *ad Infinitum*; so that what we take of A becomes nearer and nearer, *ad Infinitum*, equal to A; and consequently the Relation of these Numbers becomes nearer, *ad Infinitum*, to a true Expression of the Relation of these two Magnitudes, A, B; for they express the Relation of two Quantities, one of which is equal to B, and the other equal to A, infinitely near, or within less than any assignable Difference: Which shews us also this remarkable Truth, that what hinders any two Magnitudes to be perfectly Commensurable, is a Magnitude infinitely little or less than any assignable one; which being neglected as nothing comparatively, the two Quantities are Commensurable, infinitely near.

In this Manner then all Quantities are reduced to the Relations of Number to Number, by distinguishing Relations into such as are accurately so, and such as are so infinitely near. And the valuable Use and Application of this is, that whatsoever Conclusions can be drawn from the Proportion of Numbers, the same must hold true in the Proportion

portion of Incommensurables, since they are infinitely near as Number to Number: So that whatever four proportional Quantities these Letters represent A : B :: C : D, we may argue with them the same Way as if they were Numbers. For since there are Numbers which express any two Quantities by equal Parts, either accurately or infinitely near, the Conclusion can never be false, while these Expressions remain indefinite as to Numbers: Since whatever Error might be in supposing them determinate to any certain Degree of Approximation, it is corrected by supposing the Approximation carried further on; and because this can be done without End, the general Conclusion from these indefinite Expressions must be accurately true. So that the same Demonstrations reach to all Kinds of Magnitudes, Commensurable and Incommensurable; and by this Means the universal Doctrine of Proportion is reduced to the Science of Arithmetick.

2°. There is also another, and perhaps, a better Way of conceiving and expressing the Relations of Quantities Incommensurable. Thus; Suppose two such Quantities, A and B; if B, the greater, is divided by A, let the Quote be q, and the Remainder m; so that B contains A, q Times, and the Quantity m over; which is Incommensurable to B (for else A and B would be Commensurable:) Again, we can conceive A divided into a Number of equal Parts, each of which is a lesser Quantity than m; so that m contains a certain Number of these Parts, or is equal to a certain Fraction of A, with a Remainder n, Incommensurable to m; consequently B contains A, q Times, and that certain Fraction of a Time, with the Remainder n over: In like Manner we can conceive n equal to a certain Fraction of A, with an Incommensurable Remainder o; so that B contains A, q Times, and the Sum of those two certain Fractions of a Time, with the Remainder o over. In this Manner we may proceed *ad Infinitum*, considering the last Remainder as a certain Fraction of B, with a new Incommensurable Remainder, still decreasing infinitely; so that B is equal to q Times A, and the Sum of that infinite Series of Fractions of a Time, (*i. e.* of A.) Therefore B may be expressed by Ar; r representing the Sum of q, and that infinite Series of Fractions. Now, if the greater of two other Incommensurables being divided by the lesser, there arises the same Quote q, and also the same Series of Fractions, by dividing the lesser and the Remainders in the Manner above mentioned; then the Relation is the same; so that the lesser being called B, the greater is Br; and these are : : *l*, A : Ar : : B : Br. Thus then we have an universal Method of representing all Quantities and their Proportion: For whatever Quantity A represents, Ar will represent another greater; which is either *Commensurable* to it, if r is a determinate Number, Integral or Fractional, or *Incommensurable*, if r expresses a Number mixt of a whole Number, and an infinite Series of Fractions decreasing. Or A may be the greater of the two, and Ar express the lesser: In which Case r will represent either a certain determinate Fraction, or infinite Series of Fractions decreasing; and in both Cases, that Series carried, *ad Infinitum*, can never be equal to Unity.

Thus also we see the universal Doctrine of Ratios reduced to Arithmetick, under the Distinction of determinate and indeterminate Ratios; whose Equalities constitute *Proportion*; and being expressed in a general and uniform Manner (as A : Ar : : B : Br.) The Conclusions drawn from the Equality of r (whatever this is in it self) are alike true and good.

I shall finish this Section with a few Consequences from the Nature of Commensurable and Incommensurable Quantities, and their Arithmetical Expressions.

1. If A : B : : C : D, the one Ratio being determinate or surd, so is the other, because they are equal. Or, as A : B are Commensurable or Incommensurable, so are C : D.

2. Quan-

2. Quantities A, B, that are both Commensurable to the same Quantity C, are Commensurable to one another.

3. If A is Commensurable to C, and B Incommensurable to C, then A, B are Incommensurable; for if they were Commensurable, then C and B were also Commensurable, by the last, contrary to Supposition.

4. If A, B are Commensurable, they are both Commensurable, or both Incommensurable to the same, C; for supposing A Commensurable to C, so is B, by the second.

5. As A, B are Commensurable or Incommensurable, A+B is so to them both. And if A+B is Commensurable or Incommensurable to A or to B, it is so also to the other; and so also is A to B.

SCHOL. A certain or determinate Number, and a Surd, are in the true absolute Sense Incommensurable; yet there is in Arithmetick another more limited Sense of Commensurability and Incommensurability, which is also among determinate Numbers, and depends upon their having or not having another common *aliquot* Part but Unity; though all these are Commensurable in the absolute Sense. The Theory of Numbers depending upon this limited Distinction of Commensurable and Incommensurable, which is very considerable, you have in the next Book; in the mean Time we pass on to another Consideration of Ratio's.

§. 2. *Concerning the* Arithmetick *of* Ratios.

AMong Authors there are some who talk of Ratios as a particular Kind of Quantities different from pure Numbers; and hence they ascribe to them the common Affection of Quantity, *viz.* a Capacity of more and less, or of Increase and Decrease: But as they imagine them to be of a Nature essentially different from pure Numbers, either Integers or Fractions, so they pretend to an Idea of the Addition of Ratios, and other Arithmetical Operations about them, quite different from these about pure Numbers.

How chimerical, and void of all solid and reasonable Foundation, this Notion is, the very ingenious and learned Doctor *Barrow* has sufficiently shewn in his Mathematical Lectures. I shall only mention one principal Argument of his, *viz.* That pure Relations (for it is the abstract Relation of the Antecedents being after a certain Manner contained in, or containing the Consequent, which they call the Ratio; and which Name I apply to the Exponent of the Relation) cannot be any absolute Things; else there is no Difference betwixt Things absolute and relative; which is a manifest Absurdity. He shews, that it is impossible to make any Comparison of two Ratios, unless they have a common Consequent, or be reduced to that State, *i. e.* four Quantities of one Kind must be found (or supposed) whereof the two several Couplets have the same or an equal Consequent; and their Ratios the same as those in the Question; and then to say, that the one Ratio is greater or lesser than the other, can have no other Meaning, says he, but that the one Antecedent is greater or lesser than the other, for there is no other real Thing in this Case capable of being compared as to more and less. And so the Quantity of the Relation, or Ratio, can be nothing else but the Quantity of the Antecedent, when the Ratios are reduced to a common Consequent.

Now, though the Doctor censures the groundless Notion of a real and absolute Quantity of Relations (as such) yet he allows, as useful and convenient, the received Way of speaking of Ratios being equal or unequal, provided it be taken in the only true Sense and Meaning, which has a rational Foundation, as he explains it; and which is the same Sense in Effect that I have followed in the Foundation I have laid of the Doctrine of Proportion Geometrical, in Chap. I.

All that I shall add further upon this Question, is in short this, *viz.* That as we can form no distinct particular Idea of the Ratio of any two Quantities, which are not as Number to Number; so in the Proportion of Numbers, or of any Quantities which are as Number to Number, though the abstract Relation of any Number or Quantities being equal to a certain Part or Parts of another (which is the Geometrical Relation of two Numbers, or Quantities expressed by Numbers) is no real and absolute Quantity, yet the Exponent of that Relation (which is the Thing I call the Ratio, in distinction from the pure abstract Relation it self, of which it is the Exponent) being a Fraction proper or improper, different Exponents, or Ratios, as I take the Word, are capable of more and less, and of being compared in Quantity the same Way as different Fractions are; *Thus*, for *Example*, if A is $\frac{4}{5}$ of B, and C $= \frac{2}{3}$ of D; then we may say, that the Ratio of A to B is greater than that of C to D; meaning no other Thing, than that A is a greater Fraction of B than C is of D; for being reduced to a common Denomination they are $\frac{4}{5}=\frac{12}{15}$ and $\frac{2}{3}=\frac{10}{15}$, which is the same Sense of the Quantity of a Ratio that we have heard above in the Doctor's Reasoning: For these four Quantities are either four Numbers, or expressible, according to the supposed Ratios, by four Numbers, *viz.* $2:3$, and $4:5$; and by reducing the Fractions, or Exponents of the Relations, to a common Denominator, which is reducing the Ratios to a common Consequent, they are $10:15$ and $12:15$; and from the Comparison of the Antecedents 10 and 12, the Comparison of the Fractions or Ratios is made, and their Quantities determined.

Observe also, That we don't compare $\frac{4}{5}$ of B to $\frac{2}{3}$ of D; for if B, D are Things of different kind, no such Comparison can be made; and though they were of one Kind, yet $\frac{2}{3}$ of D might be greater than $\frac{4}{5}$ of B, according as the Quantities of B and D happen to be. But we simply compare the two abstract Fractions $\frac{4}{5}$ and $\frac{2}{3}$; which express the Relations without regard to any Difference or Likeness of the Things; because the immediate Subject of the Relation is mere Quantity; which requires no more, but that the Terms of the Relation be of one Kind in each Couplet; and then it is plain, That A's being $\frac{4}{5}$ of B is being a greater Part of it, than C's being $\frac{2}{3}$ of D is of it; whatever kind of Things A, B, and C, D are.

If the Authors, or their Followers, whom *Barrow* opposes, were content with this Notion of the Quantity of a Ratio, then they should be forced to own, that the Arithmetick of Ratios is in all respects coincident with that of Fractions (which are relative Numbers, or Expressions of Quantities relatively to others) and so they would have no Reason to talk of a new Species of Quantities; or of any different Notion of Addition, and other Operations about these Quantities. Some of these Authors, however abstractly they pretend to think about Ratios, as Quantities of a particular Species; yet as they represent them no other way than as Quotes of the Antecedent divided by the Consequent (which are Fractions, or reducible to such) so they perform all Operations with them as Fractions: But there are others who form a quite different Notion of the Arithmetick of Ratios; and though the Ground of the Application is perhaps arbitrary and whimsical, yet as it is capable of being propos'd in other

more

more reasonable Terms, and is, indeed, no other but the Application of some of the preceding Propositions, I shall here briefly explain it.

I. *Of the Addition of* Ratios.

Suppose any Numbers, $a : b : c : d : e$, containing any Ratios; the Ratio of the Extremes $a : e$ is, by the Authors I have now in view, called the Sum of the intermediate Ratios, *viz.* $a : b$, $b : c$, $c : d$, $d : e$; for no other Reason, I know, than that they are continued into one Series, and so exhibit a certain Kind of Adjunction of the Ratios. This Operation or Problem is plainly then no other than this, *viz.*

Having two or more Ratios given, to find the Ratio of the Extremes of a Series, whose intermediate Terms are in the given Ratio; the Solution of which see in *Coroll.* IV. *Probl.* I. Chap. IV.

Exam. To add $2 : 3$ and $4 : 5$ the Sum is $8 : 15$; for $2 : 3$ and $4 : 5$ continued make this Series $8 : 12 : 15$; and by the Rule referred to, the Sum or Ratio sought is thus found, *viz.* $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$: So that, in the last Place, you may observe, that this Addition of Ratios is the same as Composition of Ratios; and is Multiplication and not Addition of Ratios considered as Fractions.

Schol. Though in the true and proper Notion of the Addition of Quantities, the Sum is greater than any of the Parts; yet in this Application the Sum will always be a lesser Ratio, considered as a Fraction, than any of the Parts, if they are all proper Fractions; but it will be greater when they are improper Fractions. And in the last Place, if we consider only the Distance of the Terms in a Series of Numbers, then the Extremes being at greatest Distance, in this respect, the Ratio of the Extremes is the greatest; but to take it in this Manner is a very fantastical Notion of its Quantity.

II. *Of the Substraction of* Ratios.

This depends upon the former, and is to be conceived thus: Let there be any three Numbers $a : b : c$; then as the Ratio $a : c$ is called the Sum of $a : b$ and $b : c$, so either of these is call'd the Difference betwixt the other and $a : c$; and so is reducible to this Problem, *viz.*

Having two Ratios, to find a third, which being continued into one Series, with one of the given Ratios, the Extremes shall be in the other given Ratio. The Solution of which is opposite to that of Addition, and is therefore done by dividing the Substrahend as a Fraction, by the Substractor.

Exam. To substract $2 : 3$ from $8 : 15$, the Remainder is $4 : 5$; for $\frac{8}{15} \div \frac{2}{3} = \frac{24}{30} = \frac{4}{5}$, as this Series shews; $16 : 24 : 30$; where $16 : 24 :: 2 : 3$; $24 : 30 :: 4 : 5$, and $16 : 30 :: 8 : 15$; or this Series, $24 : 30 : 45$; where $30 : 45 :: 2 : 3$, and $24 : 45 :: 8 : 15$. From all which it's easy to observe, that this is the Effect of reducing the two given Ratios to one common Antecedent or Consequent (by *Probl.*II.Ch.IV.) the Ratio of the other two Terms found, being what is here called the Remainder or Difference, in the Sense opposite to the Sum in Addition.

III. *Of the Multiplication of* Ratios.

Case 1. The Multiplier being an Integer, this is nothing but a repeated Addition of the ſame Ratio, or the Continuation of the ſame Ratio to a given Number of Terms; and ſo reſolves into this, *viz.*

Finding the Ratio of the Extremes of a Series $\div l$, *which is in a given Ratio, and the Number of Terms* 1 *more than a given Number (which is the Multiplier.)* Which is the Application of *Cor.* IV. *Prob.* I. Ch. IV. Thus, to multiply the Ratio $a:b$ by 3, the Product is $a^3:b^3$; and univerſally $a:b$ multiplied by n is $a^n:b^n$. For the Extremes of a Series $\div l$ in the Ratio $a:b$, and whoſe Number of Terms is $n+1$, (or the Number of whoſe intermediate Ratios is n) is by *Cor.* IV. *Probl.* 1. $a^n:b^n$.

Exam. The Ratio $2:3$ continued to 5 Terms is $16:24:36:54:81$. And ſo $16:81$ $(=2^4:3^4)$ is the Product of $16:24$ $(=2:3)$ by 4; becauſe this Ratio is repeated in a Series 4 Times.

Schol. A Ratio may be alſo multiplied by a Fraction (which is the ſecond Caſe) But as the multiplying by a Fraction, and dividing by its Reciprocal, are the ſame Thing, and the Operation depends upon the Diviſion by a whole Number; I muſt refer this till the firſt Caſe of Diviſion be explained; where you'll ſee both explained together.

IV. *Of the Diviſion of* Ratios.

Caſe 1. To divide a Ratio by a whole Number. As Diviſion is oppoſite to Multiplication, ſo is the Senſe and Work of the Diviſion of Ratios to that of Multiplication. Therefore as in a Series $\div l$, $a:b:c:d:e$, the Ratio of the Extremes $a:e$ is called the Product of the common Ratio $a:b$ by the Number of Terms leſs 1; ſo that common Ratio $a:b$ is oppoſitely the Quote of the Ratio of the Extremes $a:e$ divided by the Number of Terms leſs 1; ſo that this Work reſolves into this Problem, *viz.*

Having one Ratio given, to find another, which being continued into a Series $\div l$ *of a given Number of Terms* (viz. 1 *more than the propoſed Diviſor*) *the Extremes ſhall be in the given Ratio.* The Solution of which is plainly had by *Coroll. Theor.* VI. Ch. IV. *viz.* finding the ſecond Term of a Series $\div l$, whoſe Number of Terms is $n+1$ (n being the propoſed Diviſor) and whoſe Extremes are the Terms of the given Ratio. For this ſecond Term with the adjacent Extreme, or that Term of the given Ratio which we chuſe to call the firſt Term of the Series, contain the Ratio ſought.

Exam. The given Ratio, $16:81$, and the Diviſor 4, the Ratio ſought is found by the *Coroll.* referred to, $2:3$, which is equal to $16:24$; as in the Series $16:24:36:54:81$.

Caſe 2. Both for Multiplication and Diviſion. To multiply a Ratio by a Fraction, or divide it by the reciprocal Fraction. Let any Series $\div l$, be $a:b:c:d:e:f$; then as $a:f$ is the Product of $a:b$ by 5, (the Number of Terms leſs 1) and $a:b$ the $\frac{1}{5}$ of $a:f$; ſo $a:c$ is $\frac{2}{5}$, and $a:d$, $\frac{3}{5}$ of $a:f$; or $a:f$ is $\frac{5}{2}$ of $a:c$, and $\frac{5}{3}$ of $a:d$; *that is*, $a:c$ is the Product of $a:f$ by $\frac{2}{5}$, or the Quote of $a:f$ by $\frac{5}{2}$. Wherefore, as Multiplying by a Fraction is a mix'd Operation of multiplying and dividing by an Integer; ſo this Caſe is ſolved by applying the former Caſes of multiplying and dividing Ratios by a whole Number. Thus univerſally, to multiply by $\frac{n}{m}$ (or divide by $\frac{m}{n}$) multiply by n, and then divide the Product by m; or firſt di-

vide by m, and then multiply by n, (which will make the Work easier.) For 'tis plain, that the Ratio $a:f$, divided by 5, gives $a:b$; and this multiplied by 2, gives $a:c$, the $\frac{2}{5}$ of $a:f$. Or, multiply $a:f$ by 2, and call the Product $a:l$; then is $a:f::f:l$; therefore betwixt $f:l$ there fall 4 Means in the same Ratio, as that of $a:b$, making in all this Series, $a:b:c:d:e:f:g:h:i:k:l$. Now here $a:c:e:g:i:l$ are $\div\!\!\div l$; therefore $a:c$ is the $\frac{1}{5}$ of $a:l$, *that is*, the $\frac{1}{5}$ of double of $a:f$; the same as the Double of $a:b$, which is the $\frac{1}{5}$ of $a:f$. And this shews the Advantage of beginning first with the Division.

For a particular *Example*. Suppose $8:27$ multiplied by $\frac{2}{3}$ (or divided by $\frac{3}{2}$) the Product is $4:9$; as in this Series $8:12:18:27$, where $8:12::2:3$, and $8:18::4:9$; and as $2:3$ is $\frac{1}{3}$ of $8:27$, so $4:9$ is $\frac{2}{3}$ of it: But had we first multiply'd, the Question would have appear'd in this Series, $64:96:144:216:324:486:729$; wherein $64:729::8^2:27^2$, the Double of $8:27$; and $64:144$ (or $4:9$; for $64:144::4:9$) is the $\frac{1}{3}$ Part of it; because $64:144:324:729$ are $\div\!\!\div l$. To conclude,

The Sense of this Case may be resolved into this Problem, *viz. To find the Ratio of the Extremes of a Series $\div\!\!\div l$, containing a given Number of Terms*, (*viz.* 1 more than the Numerator of a given Multiplier, or the Denominator of a Divisor) *and whose Ratio is such, that another Series may be found in the same common Ratio, whose Number of Terms is another Number given*, (*viz.* 1 more than the Denominator of the given Multiplier, or Numerator of the Divisor) *and whose Extremes are in a given Ratio*, (*viz.* that propos'd to be multiplied or divided.) The preceding Series and Explication shew manifestly that this is the true and proper Meaning of multiplying or dividing a Ratio by a Fraction.

Case 3. To divide one Ratio by another, both of one Species, *i. e.* having two Ratios of one Species given, to find how oft the one is contained in the other; or to find that Number by which the one being multiplied (according to *Case* 1, *Multiplication of Ratios*) the Product shall be equal to the other given Ratio; and if the Divisor is not an *aliquot* Part of the Dividend, we are to find the greatest Number of Times it is contained in it, and also the Ratio that remains over.

Rule. Substract the Divisor from the Dividend (by Substraction of Ratios) and the same Divisor from the Remainder, and the same Divisor again from the last Remainder, and so on continually, till the Remainder be a Ratio of Equality; and then the Number of Substractions is the Number sought; in which Case the Divisor is an *aliquot* Part of the Dividend: Or, till the Species of the Ratio in the Remainder is different from that of the given Ratios; and then the Number of Substractions less 1 is the greatest Number of Times the Divisor is contained in the Dividend; and the last Remainder but one, is the Ratio that is contained in the Dividend more than so many Times the Divisor.

Thus, if the Series $a:b:c:d:e$ is $\div\!\!\div l$, then is $a:b$ contained in $a:e$ four Times: For $a:b$ taken from $a:e$, leaves $b:e$; and from this taking $a:b$ (or $b:c$) the Remainder is $c:e$; and from this take $a:b$ (or $c:d$) the Remainder is $d:e$; and from this take $a:b$ (or $d:e$) the Remainder is $1:1$; and the Number of Substractions being 4, this is the Quote.

Again:

Again: Suppose $a : b : c : d$ are $\div l$, but $d : e$ a different Ratio; then supposing the Series to increase from a to e; 'tis plain that $d \div e$ must be greater than $c \div d$; or e must be less than a third Proportional to $c : d$; for if it were greater, we should have another Term betwixt d and e in the preceding Ratio. Now then, $a : b$ from $a : e$, leaves $b : e$ of the same Species. Again, $a : b$ (or $b : c$) from $b : e$ leaves $c : e$ of the same Species; $a : b$ (or $c : d$) from $c : e$, leaves $d : e$ of the same Species. Lastly, $a : b$ from $d : e$ makes the Remainder $db : ae$, which is of a different Species; for $a : b$ and $d : e$, being both Ratios of the lesser to the greater; and $\frac{a}{b}$ a proper Fraction lesser than $\frac{d}{e}$, the Quote $\frac{db}{ae}$, is an improper Fraction. If the Series decrease from a to e, then is e greater than a true third Proportional to $c : d$; so that when $a : b$ is taken from $d : e$, the Remainder $\frac{db}{ae}$ is a proper Fraction, because $\frac{a}{b}$ is in this Case greater than $\frac{d}{e}$. Wherefore the Number of Substractions, less 1, *viz.* 3, is the Quote; and the Remainder of the Division is the Ratio $d : e$.

BOOK V.

Containing these following SUBJECTS,

viz.

I. Of the *Composition* and *Resolution* of *Numbers*.
II. Of *Figurate Numbers*.
III. Of *Infinite Series*.
IV. Of *Infinite Decimal Fractions*.
V. Of *Logarithms*.
VI. Of the *Combinations of Numbers*.

CHAP. I.

Of the Composition and Resolution of Numbers: Or, The Doctrine of Prime and Composite Numbers; with that of the Commensurability and Incommensurability of Numbers.

§. 1. Containing the General *Principles* and *Theory*.

DEFINITIONS.

I. ONE Number is said to *measure*, or be a *Measure* of another, when it is contained in it a certain Number of Times precisely; so that being taken out of it as oft as possible, there shall nothing remain over. Thus, 4 measures 12; because it is contained in it precisely 3 Times. *Observe* also, that one Number is said to measure another by that Number which is the Quote: So 4 measures 12 by 3; and reciprocally, 3 measures 12 by 4: And hence any Number with the Quote, by which it measures another, may be called the reciprocal Measures of that Number.

COROLLARIES.

1. Every *aliquot* Part of a Number measures it; and every Number which measures another, is an *aliquot* Part of it.

2. Unity measures every Number by that Number it self; and every Number measures it self by Unity; and these are the greatest and least Measures of any Numbers; which are also reciprocal Measures.

II.

II. A Number is called the *Common Measure* of two or more Numbers, when it measures each of them: So 3 is a Common Measure of 6, 9, 12. And if it's the greatest Number that measures them, it is called their *Greatest Common Measure*; as Unity is their least.

III. A Number is called a *Prime Number*, which has no Measure, but it self and Unity; as 2, 3, 5, 7; and which consequently is the Product of no other Numbers.

IV. A Number is called a *Composite Number*, which has some Measure besides it self and Unity; and which consequently is the Product of some two other Numbers: For every Measure has its Reciprocal, and their Product is the Number measured by them, from the Nature of Division. Thus, 3 measures 12 by 4, and 3 Times 4 is 12.

V. Two or more Numbers are said to be *Commensurable*, when they have some common Measure besides 1. Thus, 6 : 9 are Commensurable, because 3 measures them both; and 5, 10, 15, because 5 measures them all.

VI. Two or more Numbers are said to be *Incommensurable*, when they have no common Measure besides 1, as 3, 4; or 4, 5, 6. Such Numbers are also said to be *Prime* to one another, or among themselves; though none of them be really *Prime* in it self.

Corol. Two or more Prime Numbers are *Incommensurable*, because they have no Common Measure but 1. And hence again; if several Numbers, A, B, C, &c. are Commensurable, no two of them can be Prime Numbers; and if one of them is a Prime, it must be the common Measure of the Whole, else they have no Measure, since that Prime has no other Measure besides 1.

Schol. Tho' Unity is a common Measure of all Numbers, yet the Notion of *Composition* and *Commensurability* is limited so as to exclude 1 from being a Measure: For since 1 measures all Numbers, if this were admitted, there would be no such Distinctions as Prime and Composite, Commensurable and Incommensurable. If we take *Composition* in the largest Sense, then Unity is the only Number which we can call *Simple*; all others being Collections or Compositions of Units: But this Consideration is too general and simple to be of any Use in discovering the Properties of Numbers; and therefore the Composition here treated of, is that particular Kind which depends upon Multiplication, taken in its more proper and strict Sense, as applied to the Repetition of Numbers, or multiplying them by a Number greater than Unity; because Unity apply'd as a Multiplier, makes no Alteration of the Number to which it is apply'd.

Observe also, That *Integral* Numbers only make the Subject of this Chapter: For if Fractions were admitted, then there is no Number, either Integral or Fractional, but some Fraction will measure it. For Example; Let A be any whole Number; take any Fraction whose Numerator is 1, as $\frac{1}{n}$, it will measure A; the Quote being nA. Again, let $\frac{A}{B}$ be any Fraction, 'tis measurable by this Fraction $\frac{A}{Bn}$, the Quote being $\frac{ABn}{AB}=n$. Again *observe*, that the Distinction of Fractions into Simple and Compound, explained in Book II. is nothing like this Distinction of Prime and Composite; even though Multiplication is concerned in that Composition; for that is merely a Distinction of two different Forms of expressing the same Quantity: Thus, $\frac{2}{3}$ of $\frac{4}{5}$, and $\frac{8}{15}$, are but the same Thing differently conceived and expressed: So that if we take the Notion of Composition in General, as the Effect of Multiplication,

tiplication, then $\frac{8}{15}$ is a compound fractional Number, tho' expressed in a simple Form: And in this Sense we can call that only a Simple Fraction which is not the Product of two real Fractions, proper or improper (excluding such improper ones whose Value is an Integer, and not a mix'd Number) and such only are all Fractions, whose Denominators are Composite Numbers. Example: $\frac{4}{5}$ is a Simple Fraction, both in Form and in its Nature; because no two Fractions produce it, or it is not the Fraction of a Fraction; but this $\frac{7}{15}$ is Simple in its Form, and in its Nature Compound; for it is $\frac{7}{5}$ of $\frac{1}{3}$, and $\frac{8}{15}=\frac{2}{5}$ of $\frac{4}{3}$, or $\frac{4}{5}$ of $\frac{2}{3}$: And this Simplicity and Composition is what belongs to Fractions, but has nothing to do in the Subject of this Chapter.

VII. A Number is called *Even*, which is Measurable by, or is a Multiple of 2; as 2, 4, 6, &c.

VIII. A Number is called *Odd*, which is not Measurable by, or is not a Multiple of 2; as 3, 5, 7, &c.

COROLLARIES.

1. An odd Number divided by 2, leaves 1 of Remainder.

2. Take the natural Progression 1 . 2 . 3 . 4, &c. and beginning at 2, take every other Number, *i. e.* take one and leave the next continually, and so you have the whole Series of *even* Numbers, 2 . 4 . 6 . 8 . 10, &c. for the Series, 1, 2, 3, &c. having 1 for the common Difference, the Difference of any Term, and the next but one, is 2; consequently beginning at 2, and taking every other Term, we have a Series differing by 2; which beginning with 2, is therefore the Series of Multiples of 2, *i. e.* of all *even* Numbers. Hence again, if we begin at 3, and take every other Term, as 3, 5, 7, 9, &c. we have the Series of *odd* Numbers; which proceeds also by the common Difference of 2.

3. If we take the natural Progression, 1 . 2 . 3, &c. and double each Term of it, the Series of Products is the Series of *even* Numbers; because it is the Series of Multiples of 2. And taking the same natural Progression, if we take the Sums of every two adjacent Terms, thus, $1+2 : 2+3 : 3+4$, &c. these make the Series of *odd* Numbers 3 : 5 : 7, &c. for the first, $1+2=3$ is the first *odd* Number, and the Series proceeds by the constant Difference of 2; because every two adjacent Sums have one Part common, and the other Parts are either two adjacent *odd* Numbers, or two adjacent *even* Numbers; which differing by 2, therefore the Series of Sums differ by 2; and because the first is 3, they must make the Series of *odd* Numbers.

4. 1 added to any *even* Number or substracted from it, makes the Sum or Remainder the next greater or lesser *odd* Number; and 1 added to or substracted from any *odd* Number, makes the Sum or Difference the next greater or lesser *even* Number. Again, 2 added to or substracted from any *even* or *odd* Number, gives the next greater or lesser Number which is also *even* or *odd*.

5. All *even* Numbers have 2, 4, 6, 8, or 0, in the Place of Units if they exceed 8; for they proceed from the continual Addition of 2 to it self, and to every succeeding Sum; but the first of them are these 2 . 4 . 6 . 8 . 10, and consequently the same Figures must circulate continually in the Place of Units. Again, all *odd* Numbers above 9, have in the Place of Units, one of these Numbers, 3, 5, 7, 9, or 1; for all *odd* Numbers proceed from the constant Addition of 2, first to 1, and then to the Sum, making the first 5 *odd* Numbers these, 3, 5, 7, 9, 11,; whence it's plain, that the same Figures will continually circulate in the Place of Units.

5. All

5. All *even* Numbers, except 2, are *Composite*. But of *odd* Numbers some are *Prime*, as 3, 5, 7, and some *Composite*, as 9, 15, 21. And since the two Series of *even* and *odd* Numbers comprehend all Numbers, it follows, that,

6. All *Prime* Numbers are *odd*, except the *Prime* 2.

Schol. An *odd* Number may measure an *even*, as 3 measures 12: But an *even* cannot measure an *odd*. Also the Product of two *even* Numbers, or an *odd* and *even*, is always *even*; as the Product of two *odd* is *odd*; the Reasons of which you will learn afterwards. And upon these Things are founded the following Definitions; whereby all *Composite* Numbers are divided into *evenly even*, *oddly even*, and *oddly odd*. Thus:

IX. An *even* Number is called *evenly even*, which an *even* Number measures by an *even* Number, or is produced by two *even* Numbers, as $12=2\times6$, and $24=4\times6$.

X. An *even* Number is called *oddly even*, which an *odd* Number measures by an *even*; or is produced by an *odd* and *even*, as $18=3\times6$.

XI. An *odd Composite* Number is called *oddly odd*; because an *odd* Number measures it by an *odd*, as $15=3\times5$; or it is produced by 2 *odd* Numbers.

Observe, Because no *even* Number measures an *odd*; therefore *odd* Composites are but of one singular Species, *viz.* *oddly odd*; therefore to call an *odd* Number *Composite*, implies *oddly odd*; but of *even* Numbers there is a Variety: Also besides the preceding two general Distinctions, it's remarkable, that some of them are *evenly even* only, *i. e.* they are not also *oddly even*; as $8=2\times4$, which no *odd* Number can measure. Some of them are *oddly even* only, *i. e.* which are not the Product of two *even* Numbers, as $14=2\times7$. Lastly, some are both *evenly* and *oddly even*, as $12=2\times6=3\times4$.

Again, *observe*, That though 1 may answer to the general Definition of an *odd* Number; yet it's excluded in all that follows especially in what relates to the three last Definitions; because these Names imply Composite Numbers, in which 1 is no Component Factor in a proper Sense. It's true indeed, that if we apply 1 as an *odd* Number, in some of the following *Theorems* they will still be true; but then it is to no Purpose, because they will coincide with some other Thing.

XII. A Number is called *Perfect*, which is equal to the Sum of all its *aliquot* Parts; as $6=3+2+1$, which are all the *aliquot* Parts of 6.

XIII. A Number is called *Abundant*, the Sum of whose *aliquot* Parts exceeds it; as 12, whose *aliquot* Parts are $1+2+3+4+6=16$.

XIV. A Number is called *Deficient*; the Sum of whose *aliquot Parts* is less than it; as 8, whose *aliquot* Parts are $1+2+4=7$.

AXIOMS.

1st. If a Number, A, measures each of the several Numbers, B, C, D, *&c.* it will measure their Sum. And if it measure them all but one, it cannot measure the Sum.

2d. The Number A, which measures the Sum of two Numbers, B+C, if it measures one of these Numbers, it will measure the other also; or, if it measures the Sum of several Numbers, and also each of the Parts to one, it must measure that one also.

Corol. The Sum of two Numbers is Commensurable with each of them; or it is Incommensurable with each of them; and cannot be Commensurable with the one, and Incommensurable with the other: And the Commensurability or Incommensurability of the Sum with each of them, is according as they are to one another Commensurable or Incommensurable; and reversely, as the Sum is Commensurable or not to each of them, so are they Commensurable or not to one another.

3d. The

3*d*. The Number A, that measures another, B, measures all the Numbers that B measures, *i. e.* all the Multiples of B; and what is measured by A, is so by all the *aliquot* Parts of A.

COROLLARIES.

1. No *even* Number can measure any *odd* Number; for then 2, which measures all *even* Numbers, would measure an *odd* Number, contrary to the Definition of an *odd* Number.

2. If a Number, A, is a Multiple or *aliquot* Part of another, B, which is a Multiple or *aliquot* Part of another C, which is so of another, D, and so on; then is each lesser of these Numbers, a Multiple or *aliquot* Part of all the greater, *i. e.* A of B, C, D, &c. B of C, D, &c.

3. From this and *Ax.* 2. follows, that if one Number, A, measures each of two other Numbers, B, C, it will also measure the Remainder after B is taken out of C, as oft as possible.

4*th*. Of whatever Factors any Number is compos'd by Multiplication, it is resolvable into the same by Division; *i. e.* it is measurable by each of these Factors, or the Product of any two or more of them; and the Quote is always the Product of all the rest of them: Thus, if $N = a \times b \times c \times d$, then $N \div a = bcd$, and $N \div ab = cd$.

Schol. A Number may be distributed into Parts, so that though another Number can measure none of these Parts; yet it may measure the Whole: But if it measures the Whole, it's always possible to separate it into Parts, each of which that Number will measure. Again, if a Number measures none of the Parts, or not all of them, and yet measures the Whole, it must measure the Sum of all the Remainders that happen upon the Division of the several Parts; for these being taken away, it measures what remains; and measuring the whole, therefore it measures the Sum of these Remainders, which is the other Part of the Whole.

PROBLEM I.

Of all the *odd* Numbers, not exceeding a given one, to distinguish which of them are *Prime*, and which *Composite*; and consequently to find whether any *odd* Number is *Prime* or *Composite*.

Rule. Begin with the Number 3, and take the Progression of *odd* Numbers 3, 5, 7, 9, 11, &c. till you have the given Number: Then beginning at 3, the first *Prime odd* Number, set some Mark, as a Point or Dash over the third Term after it, and over the third Term after this, and so on till you have not 3 Terms within the given Limit. Then begin at 5, and number 5 Terms after it, setting the same Mark over the 5th, and over the 5th after this, and so on continually as long as you have 5 Terms; do the same from all the following Terms 7, 9, &c. till you come to one after which you cannot find as many Terms as it expresses. And observe that where, in the Course of the Work, you find a Mark already, you need not put a new one. The Numbers thus marked, are all *Composites*, the others not marked being all *Primes*.

Exam. To find all the *Primes* from 3 to 79, they are these; 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79; as you see them marked in this Scheme:

$$\begin{array}{ccccccccccccccccccccc} 3 & 5 & 7 & \dot{9} & 11 & 13 & \dot{15} & 17 & 19 & \dot{21} & 23 & \dot{25} & \dot{27} & 29 & 31 & \dot{33} & \dot{35} & 37 & \dot{39} & 41 \end{array}$$

$$\begin{array}{ccccccccccccccccccccc} 43 & \dot{45} & 47 & \dot{49} & \dot{51} & 53 & \dot{55} & \dot{57} & 59 & 61 & \dot{63} & \dot{65} & 67 & \dot{69} & 71 & 73 & \dot{75} & \dot{77} & 79 \end{array}$$

Demon-

DEMON. Since an odd Number cannot be measured by an even (*Coroll.* I. *Ax.* IIId.) therefore if an odd Number is composite, it is a Multiple of some lesser odd Number by some other; and hence 'tis plain, that if we can distinguish all the odd Numbers within the Limits of the Question, which are the Multiples of each odd Number by every other, we have all the Composites within the Limits of the Question. Now that these are truly found by the Rule, I thus prove:

The common Difference in the Series of odd Numbers is 2; therefore a Term distant from any Term, as far as this Term expresses (*i. e.* the third Term after 3; or fifth after 5, *&c.*) is the Multiple of that former Term by 3; for it exceeds that former by as many Times 2, as its Distance from that former, *i. e.* by the Multiple of that former by 2, or of 2 by that; and consequently it is 3 Times that former. Going one Period further, according to the Rule, the next Term we mark exceeds the Term last marked, by the same Difference as it does the first Term, *i. e.* by 2 Times that first Term (because equidistant Terms taken out of an Arithmetical Progression, are equidifferent.) But the last marked is equal to three Times the first, and that now marked exceeds the preceding by two Times the first, and therefore it is equal to $3+2$, or 5 Times the first: For the same Reason the next marked will be $5+2$, or 7 Times the first; and so on in the Progression of odd Numbers, *i. e.* the several Terms marked in numbering from every Term, are the Multiples of this Term by all the Terms of the odd Series from 3. Wherefore we have found all the Multiples, not exceeding the Limits of the Question, of every odd Number by every odd Number, *i. e.* all the odd composite Numbers required; and consequently what are not marked are all prime.

SCHOLIUMS.

That the Rule and Demonstration of this Problem might not be too embarassed and difficult, I have left some Things to be explained here, by which the Work is made easier.

1st. Of the Series of Composites numbered from 3, mark the second, which is 15, with a double Point or Dash; then from that one begin the Numbering by the next odd Number 5, and mark the second of this new Series with a double Mark; then from this one begin the Numbering by the next odd Number 7, and so on through all the rest. These Terms doubly marked will all follow one another in order, and therefore 'tis always the last double Mark at which we begin for the next Step. Thus you see them marked in the preceding Example.

The *Reason* is this: Take any Term of the odd Series, its Multiples by each of the preceding Lesser coincide with the Multiples of each of these by this one. *Example:* The Multiples of 7 by 3 and 5 are the same as the Multiples of 3 and 5 by 7; whence it is plain, that if all the Multiples of all the Terms preceding any given one are marked, then we have so many of the Multiples of this one already marked, as do not exceed its Product into the preceding Term; and so we need only to begin at this Product in numbering by this Term. But again, it is plain, that the Product of any Term into the preceding, and its Product into the following, will have but one Composite betwixt them, *viz.* the Product of that Term into it self; therefore the Product of any Term into the preceding is the second after the Product of that preceding into its preceding; hence, *Lastly*, if we mark all the Multiples of 3, the first odd Number, and begin numbering by 5 at the second Composite from 3, we shall have all the Multiples of 5; and beginning the Numbering by 7 at the second of these numbered by 5, we shall have all the Multiples of 7; and so on.

2d. We may yet save a good deal of trouble in writing down the Series of odd Numbers, by this Method:

Suppose the given Limit be 99, write down the Numbers, 1, 3, 5, 7, 9 in one Line, reckoning each of them as simple Units; in a Column on the left Hand of these, write the Series 0, 1. 2. 3. 4. 5. 6. 7. 8. 9, reckoning each of these as so many 10's; then

	1	3	5	7	9
0					·
1			··		
2	·		·	·	
3		·	··		·
4			·		·
5	·		·	·	
6		··	·		·
7			·	·	
8			·	·	
9	·	·	·		··

draw Lines separating them, as in the annex'd Scheme: the Spaces made by the crossing of these Lines, taken in the continued Order of the odd Numbers from the first Space in the upper Corner on the left, against 0, and under 1, and numbered to the Right through that Line, and so successively through the rest of the Lines, are the Places in which all the odd Numbers, set down one after another in order from 1 to 99, would stand: And without writing them down, their Places are known thus; take the Figure in the Place of 10's of any Number in the Column on the left Hand, and the Figure in the Place of Units in the upper Line; then the Space where the Line of Spaces from the Figure in the Place of 10's, and the Column of Spaces from the Figure in the Place of Units, do meet, is the Place of that Number. Wherefore if we begin at the Space under 3 in the first Line of Spaces, and number the Spaces in order from this towards the right Hand through every Line, and mark the Spaces as before we did the Numbers represented by these Spaces, we shall have the Primes and Composites the same Way distinguished; with this plain Advantage of Ease in the Work, that we have not the Trouble of writing down all the odd Numbers separately. There are other Advantages of this Method, which you'll learn afterwards.

Again, If the Limit be 999, carry on the odd Numbers upon the upper Line to 99; and the Series 0 . 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 on the Left, represents 100's.

By this you'll easily perceive, how the Table may be made to any Limit. Afterwards (see *Probl.* III.) you'll find a Table carried to 999, with other Work upon it, to serve other useful Purposes.

3d. When you begin to number by any Term, see first if it be a Prime or Composite, which the Table will shew, according as the Place of it is marked or not; for all the Composites within the Limits assigned, which do not exceed the Product of this Term into the preceding (and in some Cases some that do exceed this Product) are already marked; and therefore if this Term is Composite it is already marked: Then if the supposed Term is a Prime, you must go on by the Rule; but if it is Composite, all its Multiples are already marked, because they are also Multiples of any of its component Parts, and all the Multiples of these are already marked: Only it will be necessary to number out the first two Periods, that the second Composite in numbering by this Term may be doubly marked, in order to know where to begin for the next Term.

4th. If it be proposed to find, Whether any given odd Number be Prime or Composite, it will not be always necessary to find all the Primes and Composites within that Limit; for if in the Course of the Work the Mark of a Composite falls upon the given Number, we have the Question solved, and there we may stop. And if you have already a Table made, there you have the Question solved for all Numbers not exceeding the Limits of the Question. Afterwards (see *Coroll. Probl.* III.) you'll see another Method of finding whether a Number be Prime or Composite, without carrying a Table so far as the given Number.

5th. The last Remark I make on this Question is, That all Numbers which have 5 in the Place of Units, as 15, 25, 35, *&c.* are Composite Numbers measurable by 5; for if that 5 in the Place of Units be taken away, what remains has 0 in the Place of Units, and so is a Multiple of 10, and consequently a Multiple of 5; therefore 5 which measures both the Parts does also measure the whole; so 5 measures 140 and 5, therefore it measures 145. Again, a Number which has any Figure, except 1, in all its Places, is a Composite measurable by that Figure; so $333=111\times3$. $7777=1111\times7$; but if 1 is

is in all its Places, the Number is Prime in some Cases, and in some Composite; so 11 is Prime, 111 is Composite, for it is $=3\times37$.

PROBLEM II.

To find if two or more given Numbers are Commensurable or Incommensurable; and what is their greatest common Measure.

Case I. *For two Numbers.*

The Rule and Reason of this Case we have already explain'd [*Book* II. *Chap* II. *Probl.* V.] where it is taught how to reduce a Fraction to its lowest Terms by first finding the greatest common Measure to its Numerator and Denominator. I shall therefore refer to that Place, and draw from it these *Corollaries.*

COROLLARIES.

1st. Whatever Number measures any two Numbers A, B, it will also measure their greatest common Measure. If you look back to the Rule and Operation referred to, this Consequence will be evident. For whatever measures the given Numbers, *i. e.* the first Dividend and Divisor, measures also the first Remainder (*Cor.* III. *Ax.* III.) and consequently it measures the second Divisor (which is the first Remainder) and the second Dividend (which is the first Divisor) and so on through all the succeeding Divisors and Dividends to the last Divisor, which is the greatest common Measure.

Exam. The greatest common Measure of 84 and 156 is 12; and because the Numbers 2, 3, 4, 6 do measure both 84 and 156, therefore they measure 12.

2d Two Numbers, whose Difference is 1, are Relative Primes, for their greatest common Measure is 1, this being the very first Remainder. And for the same Reason any Series of Numbers differing continually by 1, are relative Primes, since no two of them, whose Difference is 1, can have any other common Measure.

3d. Two odd Numbers differing by 2 are Incommensurable; for the first Remainder is 2, and the second is 1. Hence also any Series of odd Numbers differing by 2 are Incommensurable.

4th. If two Numbers, A, B, are Incommensurable, then when any Multiple of the lesser, A, as nA, is taken out of the greater, B, the Remainder B$-n$A is either 1, or a Number Incommensurable to A: For if any Number measure A, it will measure nA; and if it also measure B$-n$A it will measure B; and so A, B would be Commensurable, contrary to Supposition.

Case II. *For more than two Numbers.*

Rule. Find the greatest common Measure to any two of them; then find the same for the Number now found, and any other of the given Numbers; and again, for the last found, and another of the given Numbers; and so on, till you have gone through them all, and the last found is the Number sought.

24	40	52
	8	4

Exam. The greatest common Measure of 24, 40, 52, is 4, found thus; the greatest common Measure of 24 : 40 is 8, and that of 8 and 52 is 4.

A	B	C	D	&c.
	m	n	o	&c.

DEMON. 1°. Since m Measures A, B, and n measures m, C, therefore n measures A, B, C (*Ax.* III.) Again, o measures n, D, and n measures A, B, C, therefore o measures A, B, C, D; and so it proceeds for ever, *i. e.* each Number found in the Operation is a common Multiple to all the given Numbers.

2°. The Numbers found are the greatest common Measures of the given Numbers; for, what measures A, B measures m, and what measures m, c, measures n, (*Cor.* I.

Case I.) therefore what measures A, B, C, measures n, and consequently it is not greater than n, which is therefore the greatest common Measure of A, B, C. Again, what measures A, B, C measures n (by the last Step) and what measures n, D measures o (*Cor.* I. *Case* I.) therefore what measures A, B, C, D, measures o, and consequently is not greater than o, which therefore is the greatest common Measure of A, B, C, D.

The same Reasoning is manifestly good from one Step to another for ever; from which we have plainly gained the following Truth, *viz.*

Coroll. 5*th*. Whatever measures any Numbers A, B, C, &c. measures their greatest common Measure; so that all their other common Measures are *aliquot* Parts of the greatest.

Schol. An Integer being divided by a mixt Number less than it self may quote an Integer, and upon that Account we may say, that the mixt Number measures the other; so that a mixt Number may be the common Measure of two or more Integers. For *Example*, 18 and 24 being divided by $1\frac{1}{2}$ or $\frac{3}{2}$ quote 12, 16. But from the preceding Demonstrations we learn these Truths:

COROLLARIES.

6*th*. A mixt Number can never be the greatest common Measure of two Integers; for it's shewn, that this must be an Integer, *viz.* the last Remainder of a Division of Integers: Hence again,

7*th*. No mixt Number can be the greatest common Measure of any Number of Integers, for then it might also be the greatest common Measure of two Integers.

8*th*. Two Integers which have not another common Measure in Integers but 1, have not one of any kind, except, perhaps, an *aliquot* Fraction, (*viz.* such as has 1 for its Numerator) or some equivalent one; for 1 being their greatest Measure, no other Number can measure them, except it be a proper Fraction; but no other than an *aliquot* Fraction can do it: For suppose any other, as $\frac{a}{n}$, if it measures A, B, it measures also their greatest Measure 1, which is impossible; because the Quote of 1 by $\frac{a}{n}$ is $\frac{n}{a}$, which cannot be equal to an Integer precisely, for then $\frac{a}{n}$ would be equal to an *aliquot* Fraction, *i.e.* suppose $\frac{n}{a}=r$ or $\frac{r}{1}$, then is $\frac{a}{n}=\frac{1}{r}$, contrary to Supposition

9*th*. Integers that have a mixt common Measure, have also an Integral one greater than 1; for their greatest must be an Integer, and it must be greater than 1, because a mixt Number, which is always greater than 1, is supposed to measure them.

But *observe*; Two Numbers may have an integral common Measure greater than 1, and yet have no mixt common Measure.

Theorem I.

Every prime Number, A, is Incommensurable with every Number, B, which it does not measure.

$A=5 . B=8$ | Demon. If A, B, are Commensurable, then either A or some other Number measures them both; either of which is contrary to the Supposition.

Coroll. Of several Numbers A, B, C, &c. if one of them, as A, is a Prime, it is the common Measure of the whole, else they are Incommensurable. And if it do measure the whole, it is their greatest and only common Measure, because it self has no other Measure but 1.

The-

THEOREM II.

If a Number, A, measures one, B, of two Numbers, B, C, that are Incommensurable; it is Incommensurable with the other, C.

A=3
B=6. C=7

DEMON. If any Number measures both A and C, it will also measure B, which A doth measure (*Ax.* 3.) therefore B, C are Commensurable, contrary to Supposition.

Observe, The Reverse will not always hold; for tho' A is Incommensurable with C, it does not follow that it will measure A; because it may be Incommensurable to both A and C; as in Case A be =5.

THEOREM III.

Numbers a, b, c, &c. that are least in their Ratios, are like *aliquot* Parts of, or do equally measure, any other Numbers, A, B, C, *&c.* that are in the same Ratios respectively, *that is*, a Measures A, and b measures B, *&c.* equally. Also, the Number by which they measure them is the greatest common Measure of A, B, C, *&c.* *Reversely*, The greatest common Measure of certain Numbers A, B, C, *&c.* measures them by Numbers a, b, c, &c. that are least in the same Ratios.

	A	B	C	D
Dividends,	6	9	15	24
Divisors,	2	3	5	8
	a	b	c	d

Common Quote $q=3$

DEMON. 1°. Since, $a : b :: A : B$, and $b : c :: B : C$, *&c.* then alternately $a : A :: b : B$, also $b : B :: c : C$, *&c.* hence $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}$, *&c.*

Now if we suppose that these equal Quotes are not Integral (or if a does not measure A, *&c.*) yet because they are equal, therefore the integral Part is the same, and so also is the Fraction. Let the Remainders of the Divisions be r, s, t, *&c.* then these Fractions are, $\frac{r}{a}=\frac{s}{b}=\frac{t}{c}$, *&c.* but being proper Fractions the Numerators r, s, t, *&c.* are less than their Denominaators a, b, c, *&c.* and being equal Fractions they are respectively in the same Ratios, *i. e.* $r : s :: a : b$, and $s : t :: b : c$; hence a, b, c, *&c.* are not least in their Ratios, contrary to Supposition; and therefore a, b, c, *&c.* must measure A, B, C, equally. Again, the Quote q is the greatest common Measure of A, B, C, *&c.* for whatever it be, it measures A, B, C, *&c.* by the same Numbers which were before the Divisors, *viz.* a, b, c, &c. (from the Nature of Division.) But if A, B, C, *&c.* are divided by any Number, the Quotes are also in the same Ratio (from the Nature of Ratios.) Also the greatest common Measure must give lesser Quotes than any other Measure, and therefore either q is the greatest common Measure, or a, b, c are not the least in their Ratios: But a, b, c, &c. are least in their Ratios, therefore q is the greatest common Measure.

The *Reverse* of the Theorem is manifest from the Nature of Division.

SCHOL. Though Numbers a, b, c, d, do equally measure others, A, B, C, D, and so are like *aliquot* Parts of them, it does not follow that they are least in their Ratios; for in order to this, they must measure them by their greatest common Measure.

COROLLARIES.

1st. Numbers that are not least in their Ratios are Equimultiples of such as are so, because these are like *aliquot* Parts of the others.

2d. Here we have another Proof that a Fraction can never be the greatest common Measure of several Integers; for then the least in their Ratios would not measure other Numbers in the same Ratios, as they must do by what has now been shewn.

3*d*. We learn here how to find Numbers that are least in the same Ratios with other given Numbers, *viz.* by finding the greatest common Measure to these given Numbers, and by it dividing them, the Quotes are the Numbers sought; so the least in the same Ratios with 9 : 15 : 21, are 3 : 5 : 7, for the greatest common Measure of the former is 3, and the Quotes are 3 5. 7.

THEOREM IV.

Numbers *a*, *b*, *c*, *d*, &c. that are least in their Ratios, are Incommensurable; and the *Reverse*, Incommensurables are least in their Ratios.

a, *b*, *c*, *d*,
2, 5, 6, 7,

DEMON. If *a*, *b*, *c*, *d*, are Commensurable, then being divided by their common Measure, the Quotes will be in the same Ratios, and also lesser Numbers; therefore *a*, *b*, *c*, *d*, are not the least, contrary to Supposition.

For the *Reverse*, If the least Numbers in the same Ratios with *a*, *b*, &c. are lesser than they, then will these Numbers equally measure them by their greatest common Measure (*Theo.* I.) But *a*, *b*, &c. being Incommensurable, have no common Measure; therefore they are least in their Ratios.

COROLLARIES.

1*st*. Since Numbers being least in their Ratios, and being Incommensurable, do necessarily follow from one another, we may put any of these in place of the other in any Propositions; particularly in the preceding Theorem; so that if Numbers *a*, *b*, *c*, &c. are Incommensurable, they do equally measure any others in the same Ratios A, B, C, *&c.* and hence all the following Corollaries.

2*d*. If Numbers *a*, *b*, *c*, &c. are Incommensurable, others in the same Ratios A, B, C, *&c.* are Commensurable, and also all *Composites*; for *a*, *b*, *c*, &c. do equally measure them, and the common Quote does reciprocally measure each of them; which therefore are both Commensurable, and all *Composites*. Hence again,

3*d*. If four Numbers are : : *l*, *a* : *b* : : A : B, no three of them can be prime Numbers, nor consequently the whole; [or thus, to three prime Numbers a fourth : : *l* can't be a whole Number] for if *a*, *b*, are Primes, they are Incommensurable, and so A, B, are both Composites. Whence again, if three Numbers are : : *l*, A, B, C (or A : B : : B : C) none of the Extremes with the middle Term, nor consequently all three, can be Primes [*i. e.* to two Primes, A, B, a third : : *l* can't be an Integer] for if A, B, are Primes, therefore B, C, are Composites, so that B is both Prime and Composite; which is absurd.

In another Place (see *Theo.* XXII. *Cor.* III.) you'll see it demonstrated, that betwixt two Primes there can't be a geometrical Mean, either in whole Numbers or Fractions.

THEOREM V.

If any Numbers A, B, C, *&c.* Incommensurable, are measured by other Numbers *a*, *b*, *c*, *&c.* (*i. e.* A by *a*, and B by *b*, &c.) these last are neither in the same Ratios with the former, nor do they measure them equally, nor, *Lastly*, are they Commensurable.

DEMON. *a*, *b*, *c*, can't be in the same Ratios with A, B, C, *&c.* for since they measure them, it would follow, that they measure them equally; because if *a* : A : : *b* : B then $\frac{A}{a} = \frac{B}{b}$; and whatever the common Quote, is it will reciprocally measure A, B, C, which consequently are not Incommensurable. Again, whatever be the Ratios of *a*, *b*, *c*, if they measure A, B, C, equally, the Quote will reciprocally measure A,B,C, which

which therefore are not Incommensurable. *Lastly*, If *a*, *b*, *c*, are Commensurable, their common Measure must measure A, B, C, which they measure; consequently A, B, C, are not Incommensurable. And observe, that this last Article does also prove the first; for if *a*, *b*, *c*, and A, B, C, are both Incommensurable, they can't be in the same Ratios.

THEOREM VI.

If a Number, A, measures the Product of two others, B, C, and is Incommensurable to one of them, it will not only be Commensurable with the other, but also measure it.

DEMON. Let A measure BC by D, then is it A : B : : C : D; but if A, B, are Incommensurable (or least in their Ratio) therefore A measures C, and B measures D equally, by the last.

COROLLARIES.

1st. If a Number, A, measures the Product of two Factors B, C, (*i. e.* if four Numbers are : : *l*, A : B : : C : D) then will it either measure one of these Factors, B or C, or be Commensurable to each of them; for if it's Incommensurable to any one of them, it measures the other. But if A is a prime Number, and measures BC it will necessarily measure B or C; for if it do not measure B, it is Incommensurable to it (by *Theor.* I.) and consequently measures C, by this Theorem.

2d. If a Number, A, is Incommensurable to each of two Factors B and C, or if it's Incommensurable to the one, and does not measure the other, it cannot measure the Product (*i. e.* A, B, C, being Integers, and A : B : : C : D, then A being Incommensurable to both B and C, or Incommensurable to the one, and not measuring the other, D is not an Integer;) for if it measures the Product (*i. e.* if D be also an Integer) and is Incommensurable to one of the Factors, it measures the other, and so is not Incommensurable to it: Both contrary to Supposition. Hence again,

3d. If A is Incommensurable to B, it can't measure the Square of B, *i. e.* a third : : *l* to two Numbers A, B, that are Incommensurable, can't be found in Integers.

SCHOL. As A's being Incommensurable to both B and C is a certain Consequence of their being all three prime Numbers, so *Coroll.* 3*d. Theo.* IV. is in effect comprehended in the preceding *Coroll.* 2*d.*

Again, though three Numbers be Incommensurable (which is a different Thing from one of them being so to each of the other two) yet D may be integral, as in these 2 : 3 : : 4 : 6.

Further, in the various Circumstances of A, B, C, being all Composite Numbers, or only Commensurable, *Observe* (1°) That though they are all Composites, yet a fourth in Integers may be impossible, because A may be Incommensurable to both B and C, which is consistent with their being all Composites, as in these 4, 6, 9. (2°.) Though they are all Composites, and also Commensurable, yet a 4th Integral may be impossible; as in these, 6 : 10 : : 14, to which a 4th is 13 $\frac{1}{3}$. But to understand the general Reason of this, and what Circumstances of their Composition makes it so, depends upon some other Principles than we have yet heard, and which you will find afterwards (see *Schol.* III. *Theo.* XXIII.) where I shall shew you the general Character upon which depends the 4th's being Integral or not. *Lastly*, observe, That the three given Numbers being Commensurable, if the first of them, A, is a prime Number, then is D always an Integer; for in this Case A is the common Measure, and because it measures B and C it will also measure BC so that D, which is the Quote, is an Integer.

THEOREM VII.

The least Number A, which measures any composite Number B, must be a prime Number.

$A=3.\quad B=15$ | DEMON. If A is a Composite, then its component Parts will measure B (*Ax.* 3.) and consequently A is not the least which measures B, contrary to Supposition.

COROLLARIES.

1*st*. Some prime Number measures every Composite; or, every Composite is measured by (*i. e.* is the Multiple of) some Prime; for there must be a least Measure, and that must be prime.

2*d*. Every Number is either a Prime, or may be measured by some Prime.

3*d*. Commensurable Numbers have some prime common Measure; for their common Measure is either a Prime, or some Prime measures it, which therefore will measure these Composites: And hence also, if they have several common Measures, the least of them is a prime Number. *Reversely*, Numbers that have no prime common Measure are Incommensurable.

THEOREM VIII.

There are an infinite Number of prime Numbers: Or thus, no Number of Primes can be assigned, but another may be found different from all the given ones.

$A\times B\times C$, &c. $=P$
$P+1$; z

DEMON. Let A, B, C, &c. be any Number of Primes, whose continual Product is P, to which add 1; then if $P+1$ is a Prime, 'tis plainly different from the given ones; but if it be Composite, some Prime, as z, measures it (*Corr.* 1*st*. *Theo.* VII.) and this is a different Number from any of the given Primes; for if it be the same with any of these, then it will measure P their Product (*Ax.* 3.) but 'tis supposed also to measure $P+1$, therefore it must measure 1 (*Ax.* 2.) which is absurd.

THEOREM IX.

Take the greatest Number of Factors, a, b, c, &c. of which any Number, N, can be composed (or to whose Product it is equal) they will be all prime Numbers.

$a\times b\times c=N$
$3\times5\times7=105$

DEMON. If any of them be Composite, the component Parts of it are also Components of the given Composite N (*Ax.* 3.) and so the proposed Number of Factors is not the greatest whose Product makes N, contrary to Supposition. So if $a=n\times r$, then is $N=n\times r\times b\times c$.

COROLL. Every composite Number is equal to the Product of a certain Number of Primes, *viz.* the greatest Number of Factors by which it can be produced.

THEOREM X.

A Number, M, which is the Product of a certain Number of given Primes, a, b, c, d, &c. whether they are all different Numbers, or some of them oftner than once involved, can be measured only by one of these prime Factors, or the Product of any two, or more of them; *that is*, it cannot be measured by any other prime Number, nor by any Number which has in its Composition any other Prime, *i. e.* which is the Multiple of any other prime Number; nor, *Lastly*, by any Composite, which, though it have in its Composition no Prime different from any of these that produce M, yet has any of the same Primes oftner involved, *i. e.* is the Multiple of a greater Power of any of these Primes than what M is; as, if M has in it only the Root or Square of a, and this other has in it the Cube of a.

Exam.

Exam. $2\times3\times7\times11=462$; which Composite 462, cannot be measured by 5, which is a different Prime; nor by 15, the Multiple of 5; nor by 9, the Square of 3.

DEMON. 1*st*. That M is measurable by any one, or the Product of any Number of its own Component Primes, is plain, by *Axiom* 4. And,

2*d*. That it can be measured by none other, *i. e.* by none of these described in the *Theor.* which plainly comprehend all others, is thus demonstrated.

(1°.) It cannot be measured by any other Prime, as x.

For x is Incommensurable to a and b, because both are Primes different from x; therefore it cannot measure their Product ab (by *Cor.* 2. *Theor.* VI.) and so is Incommensurable to it (*Theor.* I.) and being also Incommensurable to another Prime c, it cannot measure the Product abc (*Cor.* 2. *Theor.* VI.) and hence is Incommensurable to it (*Theor.* I.) and being also Incommensurable to another Prime d, it cannot measure the Product, $abcd$. And so the Reasoning proceeds for ever.

Or thus also. Let d be any Prime different from x, or the Product of any Number of Primes all different from x; then, upon Supposition that x cannot measure d, I say it cannot measure the Product of one Prime more, *i. e.* dm (m being also a Prime different from x;) for if x measure dm, let it be by y, then is $x : d :: m : y$. But x being a Prime, which does not measure d, then are x, d, Incommensurable (*Theor.* I.) and so x measures m (*Theor.* IV. *Cor.* 1.) which is absurd, because m is a Prime, and different from x. But x cannot measure another Prime; therefore, by what is now shewn, it cannot measure the Product of two others, nor consequently the Product of three others, and so on, *i. e.* it cannot measure the Product of any Number of others.

(2°.) It cannot be measured by the Multiple of another Prime; for then that Prime would also measure it (*Ax.* 3.) which is contrary to the last Article.

(3°.) It cannot be measured by any Number N, which tho' it has in it no other Prime, yet has any one of the same Primes oftner involved. For suppose any one or more of them is oftner involved in N than in M; then imagine all the Prime Factors of N, that are also in M, to be taken out of both, *i. e.* let both of them be divided by the continual Product of all these common Prime Factors, and call the Quotes A, B, they will be in the same Ratio, or N : M :: A : B. But now, of those Primes that were not so oft involved in N as in M, what were more of any of them in M than in N, and what were not at all in N, will remain in B; and what were more in N than in M, will remain in A; (by *Ax.* 4.) but none of these will be in B; for because there were fewer of them in M than in N, therefore they were all taken out of M; consequently there will be some Prime in A, which is not in B, and therefore A cannot measure B; for then that Prime would measure B, contrary to what is shewn. Therefore lastly, N cannot measure M, because N : M :: A : B. And if A cannot measure B, neither can N measure M.

N) M
A) B

COROLLARIES.

1*st*. Of two Composite Numbers, A, B; if there is in the Composition of the one, any Prime which is not in the other, or any the same Prime oftner involved, these two Numbers cannot be equal: For in these Circumstances, the one cannot measure the other, and consequently they cannot be equal.

2*d*. M, the Product of a certain Number of Primes, a, b, c, d, &c. cannot be equal to (or the same Number with) N, the Product of any greater Number of Factors, whatever they be; nor to the Product of any other Choice of an equal Number of Factors; nor lastly, to the Product of a lesser Number of Factors, which are all Primes. For (1.) A greater Number of Factors are either all Primes, or are resolvable into a greater Number of Primes; and therefore, among them there must necessarily be found some Prime

different from any of these in M, or some of these oftner involved; and so N cannot measure M, and so not be equal to it: (2.) For another Choice of an equal Number of Factors, they are either all Primes, and therefore must have some different Prime, or a greater Power of some Prime; and so they cannot be equal (by the first *Cor.*); or if any of them be Composite, then being resolved into their Primes, there will be a greater Number of Factors; and so it coincides with the first Case. (3.) For the last Case, 'tis already demonstrated in the first; where it's shewn, that a greater Number of Prime Factors cannot produce the same Number as a lesser; the Reverse of which is the present Case, which we may also prove in this Manner, *viz.* A lesser Number of Primes must either have some different Prime, or a greater Power of some of the same Primes; and so N cannot measure M, and therefore cannot be equal to it: or, the Factors of N are a Part of the same Primes that compose M, and so N will be only a Part of M. Hence again reversely,

3*d*. The same Number, M, cannot be resolved into a different Number of Prime Factors. For *Exam.* It cannot be resolved into 3, and also into 4 Prime Factors: Nor into any one Number of Prime Factors, with a Variety of Choice: But every Composite has a precise limited Number of determined Primes; so that neither in the particular Primes, nor in their Number, can there be any Variety.

4*th*. Two unequal Composite Numbers may be composed, either of a different Number, or the same Number of Primes: But in both Cases these Factors are either all or part of them different Primes, or some Prime common to both, is oftner involved in the one than in the other. But then *observe*, That the lesser Composite may have either the lesser or greater Number of Factors; for that depends upon the Numbers themselves; thus, $42=2\times3\times7$; and $221=13\times17$.

5*th*. A Number, M, which is the Product of any two or more Factors, whatever they be, as $A\times B\times C$, &c. being resolved into its Primes, these can be no other than the Primes into which the Factors, A, B, C, &c. can be resolved; for else the same Number could be composed of different Primes, contrary to *Cor.* 3. And hence again, There is no Prime in the Composition of any Power, but those which compose the *Root*.

6*th*. No Numbers can measure any Power of a Prime Number, but either the Root it self, or some other of its Powers; for every other Number has in it some other Prime.

7*th*. Whatever Prime Number, N, measures any of the Powers of any Number, A, as A^n, the same will measure the Root A, and all the other Powers; for since N measures A^n, it must be one of its Component Primes, *i, e.* one of the Primes that compose A, by *Cor.* 5. therefore N measures A, and all its Multiples, or all its other Powers. Hence again,

8*th*. If any Number, N, measures A^n, and does not measure A, it's a Composite Number; for if it were a Prime, it would measure A.

Theorem. XI.

Of all the Component Primes of any Number, N, only one (if there be one) can be a Number greater than the Root of the greatest Integral Square, contained in that Number.

Demon. 1*st*. If the given Number, N, is a perfect Square, then it has no Prime in its Composition, but those that compose the Root. (*Cor.* 5. *Theor.* 10.)

2*d*. If N is not a Square, let A be the Root of the greatest Integral Square contained it; then is $\overline{A+1}^2$ greater than N: And if we take two Primes greater each than A, they must be, the one of them at least equal to $\overline{A+1}$, if not greater; and the

the other greater than this one ; consequently their Product will be greater than $\overline{A+1}^2$, *i. e.* greater than N : And therefore they are not both Component Primes of N, since they produce a greater Number.

PROBLEM III.

To find all the Component Primes of any Number.

Rule 1°. Find all the Prime Numbers, not exceeding the Root of the greatest Intergral Square, contained in it. Then,

2°. Beginning with 2, if the given Number is even ; or with 3, if it's an odd Number, try if 2 or 3 measures it, and do the same with the Quotes, as long as the same Prime measures them ; but when it does not measure, apply the next greater Prime in the same Manner ; and go on so till you have tried all the Primes, not exceeding the Root mentioned, or till you find a Quote which is a Prime Number ; then all these Primes, which were Measures to the given Number and to the succeeding Quotes, together with that Prime Quote, or the last Quote, to which there was no Measure among the Primes found, by the first Article (which is also a Prime) are the Component Primes of the given Number.

Exam. 1. To find the Primes of 42 ; the Root of the greatest Square contained in it is 6 ; and the Primes not exceeding this, are,

2, 3, 5.
2) 42
3) 21
7

And trying 42 by these, I find 2 measures it by 21 ; but this cannot be measured by 2, therefore I try 3, which measures 21 by 7, which is a Prime Number ; as is also known according to the Rule, by this, that neither 3 nor 5 measures it ; therefore 2, 3, 7, are the Component Primes of $42 = 2 \times 3 \times 7$.

2	68796
2	34398
3	17199
3	5733
3	1911
7	637
7	91
	13

Exam. 2. To find the Primes of 68796 ; the Root of the greatest Square contained in it is 262 ; and the Primes not exceeding this, are 2, 3, 5, 7, 11, 13, &c. and trying, I find 2 measures twice, 3 measures 3 Times ; 5 does not measure 637, the last Quote, by 3 ; therefore I apply 7, which measures twice, and the last Quote is 13, a Prime Number ; therefore the Component Prime Factors of 68796, are 2, 2, 3, 3, 3, 7, 7, 13.

DEMON. As there is no Matter in what Order any Numbers are apply'd by continual Division, since the last Quote will still be the same : So if certain Primes apply'd by continual Division, in whatever Order, do measure out the given Numbers, then it's plain, from the Nature of Multiplication and Division, that the continual Product of these Divisors, will again produce the same Number : And if certain Prime Factors produce a Number, no other Variety or Choice whatever of Prime Factors, can produce the same Number, by *Cor.* 3. *Theor.* 10. What remains then to be shewn is this ; that when we have got a Quote, which neither the Prime last apply'd, nor any greater, not exceeding the Root mentioned, do measure, that Quote is a Prime : The Reason is this ; none of the preceding lesser Primes can measure it ; for each of these are supposed to be taken out of the given Number as oft as possible ; and since none of the Primes, not exceeding the Root mentioned, can measure it, none of these exceeding that Root can measure it, unless it self be one of these Primes ; for if another could do it, the Quote would be a Number less than the Root, and must be either a Prime, or measurable by a Prime, which reciprocally would measure it ; consequently none greater can do it, unless it self be one of these greater Primes ; and therefore it must be a Prime Number.

Corol. By this Method we can know whether a Number is Prime or Composite, though we have a Table carried only as far as the Root of the greatest Square contained in it; for if none of the Primes of this Table measures it, then it is a Prime.

Scholiums.

1st. If a Number has 0's in the first Places on the Right-hand, cut them all off, and proceed with the remaining Figures, according to the Rule; and then among the Primes of this Number reckon as many 2's and as many 5's as the Number of 0's cut off; because $2 \times 5 = 10$; and all together are the Primes sought.

Again, if an odd Number end with 5, try it with 5 before 3; because 5 will certainly measure it, though 3 will not always.

2d. If you have a Table of Primes and Composites extending to the given Number, or to the last Quote after it is measured as oft as possible, by 2 and 5; (for which see the last Article) then seek every other Quote (which is not measurable by 2 or 5) in the Table; because this shews whether it's a Prime or Composite: So that being a Prime, you know the Work is ended; and being Composite, proceed according to the Rule.

3d. But again; we will often have the Trouble of trying Primes that do not measure the given Number, or succeeding Quotes, and which would be saved, if we knew the least Prime that measures any of these: Now this we may know by help of the Table of Primes and Composites; if the Spaces are so marked as to shew the least Component Prime of each Composite; and how this may be easily done I shall here explain. Thus:

When you are to number Spaces by any odd Number, see first whether it's a Prime or Composite (by the Table;) if it's Prime, write it in the Places of its Composites, *i. e.* in the Spaces where Points are placed by the former Method, unless some lesser Prime stand there already; as will certainly be, if the Composite belonging to that Place, has in it a lesser Prime; for then it's a Multiple of that lesser Prime, and is therefore already marked with its least Prime. *Observe* also, that with this Prime you set some other Mark, as a Point or Dash, upon the Place of the second Composite, in numbering by this odd Number, in order to know where to begin for the next.

If the Number, by which the Spaces are to be number'd, is Composite, all its Multiples are already marked; only you must number out the first two Periods, that the second Composite, in numbering by this Term, may be particularly marked (as I have done in the following Table by a Colon:) in order to know how to begin for the next.

The following Table, carried to 999, is made up in this Manner: The Use of which, for finding the Component Primes of any Number, is this;

If the Number is even, measure it by 2 as oft as possible: If it ends with 0's cut them off, and reckon as many 2's and 5's among the Primes sought; then proceed with the last Quote or remaining Number (or with the given Number, if it's odd) thus: See by the Table if it's a Prime or Composite; if Prime, the Question is solved; if Composite, you have its least Component Prime; by which, measure it, and seek the Quote in the Table; measure this by its least Component Prime, and so on till you have a Quote which is a Prime; and that Quote, with the preceding Divisors are the Primes sought.

	0	1	2	3	4	5	6	7	8	9
1			3	7		3			3	17
3			7	3	13		3	19	11	3
5		3	5	5	3	5	5	3	5	5
7			3		11	3		7	3	
9	$\overline{3}$		11	3			3			3
11		3			3	7	13	3		
13			3		7	3		23	3	11
15	3:	5	5	3	5	5	3	5	5	3
17		3	7		3	11		3	19	7
19		7	3	11		3			3	
21	3	$\overline{11}$	13	3			3	7		3
23		3		17:	3		7	3		13
25	$\overline{5}$	5	$\frac{15}{3}$	5	5	3	$\frac{25}{5}$	5	3	5
27	3			3	7	17	3			3
29		3		7	3	$\overline{23}$	17	$\frac{27}{3}$		
31			3			3		17	3	7
33	3	7		3		13	3		7	3
35	5:	3	5	5	3	5	5	3	5	5
37			3		19	3	7	11	3	
39	3			3		7	3			
41		3		11	$\frac{21}{3}$			3	$\overline{29}$	
43		11	3	7		3			3	23
45	3	5	5	3	5	5	3	5	5	3
47		3	13		3			3	7	
49	$\overline{7}$		3			3	11	7	3	13
51	3			3	11	19	3		23	3
53		3	11		3	7		3		
55	5	5	3:	5	5	3	5	5	3	5
57	3			3			3			3
59		3	7		3	13		3		7
61		7	3	$\overline{19}$		3			3	$\overline{31}$
63	3:			3			3	7		3
65	5	3	5	5	3	5	5	3	5	5
67			3			3	23	13	3	
69	3	$\overline{13}$		3	7		3		11	3
71		3		7	3		11	3	13	
73			3		11	3			3	7
75	3	5	5	3	5	5:	3:	5	5	3
77	7	3		13	3			3		
79			3			3	7	19	3	11
81	$\frac{9}{3}$			3	13	7	3	11		3
83		3			3:	11		3:		
85	5	5	3	5	5	3	5	5	3	5
87	3	11	7	3			3			3
89		3	$\overline{17}$		3	19	13	3	7	23
91	7		3	17		3		7	3	
93	3			3	17		3	13	19	3
95	5	3:	5	5	3	5	5	3	5	5
97			3		7	3	17		3	
99	3:		13	3:			3	17	29:	3

Observe, As the Numbers of the first Column on the Left are carried to 99; so the Numbers on the Head of the Table are to be reckoned as Hundreds: So 1 is 100; 2 is 200, *&c.* And for any odd Number above 99, take the Number of its Hundreds on the Head, and what in it is less than a Hundred on the Side, and in the Angle where the Lines from each meet, is the Place of that Number: So that in composing of the Table, the Numbering is along the Columns from Top to Bottom, from the first Column under 0, and so on in Order, through the rest, (which is so far different from the little Table before given, that there the Series of odd Numbers was set on the Head, and so the Numbering was along the Lines, from left to right.) If we would make a larger Table, then continue the Series of Numbers on the Head, from 9 to 10, 11, 12, *&c.* as far as you please, keeping the same Column on the Left; and reckoning these Numbers on the Head always as expressing so many Hundreds; and continue the Numbering through the Columns in Order; and so the Table may be carried to any Length: For *Example*; If the Series on the Head is carried to 99, it is 9900; and so the greatest Number is 9999. If the Series on the Head is carried to 999, it is 99900; and the greatest Number is 99999.

Again, If there is a Line over any Number, in any Space, it shews that to be the Square Root, and also the least Prime of the Number belonging to that Space. If there are two Numbers in any Space, the greater is the Square Root, and the other is the least Prime.

Observe also, That if a Number is a perfect Square, then we may find all its Primes, by a Table carried no farther than to the square Root, or the odd Number next below it, if the Number is even. For if we take all the prime Factors of the Root twice, these are the prime Factors of the Square.

Another Use that may be made of this Table, is, That by it we can easily find, whether any Number, odd or even, is a perfect Square, and what is the greatest integral Square contained in it, if it is not; thus,

1°. Suppose the Number proposed is an odd Number; seek its Place in the Table, and if it's a composite Number then find the Place next before, and also after it, that is marked with a Colon; and take the Space that is in the Middle betwixt these two (for by the Construction of the Table, the Number of Spaces betwixt them is odd, and therefore has a middle Space.) If the given Number is in that middle Space

Space, it's a square Number, because the Space next before it marked with a Colon is the Place from whence we begin to number Spaces by some odd Number, which is the Place of the Product of that odd Number by the preceding; and the next Space marked with a Colon is the Place of its Product, by the following; and therefore the middle Space betwixt them is its Square, or Place of its Product by it self. So that the middle Spaces betwixt every two marked with a Colon, are the Places of all the odd square Numbers. Wherefore if the given Number is not in one of these middle Spaces, it cannot be a Square, and the greatest Square contained in it is that belonging to the middle Space next it, towards the Beginning of the Table.

If the given Number is Prime, it's certainly not a Square, and you find the greatest Square contained in it the same way as before.

Observe, If you would find the Root of any Square found in the Table, number how many Spaces there are marked with a Colon from the Beginning of the Table, to that one next preceding the Place of that Square; number as many Terms after 3 in the Series of odd Numbers, the last of them is the Root sought; so if there are 5 pointed Spaces, the Root is 13, the 5th Term after 3, and the Root of 169. But as in making the Table, these Places of Squares are the first composite Spaces in numbering by the several odd Numbers, which are the Root of these Squares; if we not only mark these Spaces with the least Prime, but also with their Root, it will be the more convenient for this purpose; so that if any Space has two Numbers in it, the lesser is its least Prime, and the other the square Root of the Numbers belonging to that Place. And if its least Prime is also its square Root, we may either write it twice, or use some other Mark to show it, as a Line drawn over it, as I have done in this Table: And thus the Places of all odd Squares, and also their Root, are known by Inspection without any Trouble.

2°. If the given Number is even, seek in the Table the odd Number next lesser; if it's a square Number, the given Number can't be a Square, because the Difference of two integral Squares can never be 1, and that odd Square is the greatest contained in it; but if the next lesser odd Number is not a Square, seek by the Table the next odd Square; take its Root, and add 1 to the double of it; if the Sum is equal to the Difference betwixt that Square and the given even Number, then is this a square Number, whose Root is the even Number next above the Root of that odd Square: But if that Sum is either greater or lesser than that Difference, the given Number is not a Square; and if the Sum is greatest, that odd Square is the greatest Square contained in the given Number; but if the Sum is least, add it to the odd Square, and the Sum is the greatest Square contained in the given Number. The Reason of this is obvious from the Nature and Composition of Squares explained in *Book* III. particularly this, that $\overline{A+1}^2 = A^2+2\,A+1$.

PROBLEM IV.

To find all the different Numbers that measure any given Number.

Rule. 1°. Find all the Component prime Factors of the given Number by the last Problem; then,

2°. Set them all in a Line; but those that are oftner than once involved, set them down but once, and instead of the rest of them set down the Series of their superior Powers, till you have a Power whose Index is the Number of Times that that one is involved in the given Number; these are so many of the different Measures sought. And though it is in Effect the same Thing, which Prime is first set down, yet one Order may prove more convenient than another for the following Part of the Work; there-

therefore ſet down firſt all thoſe Primes, which are but once involved, and then thoſe that are oftner, with their ſuperior Powers as above, ſetting thoſe firſt that are leaſt involved; then,

3°. Beginning from the left Hand, multiply the firſt Number by the ſecond, and ſet the Product under the ſecond; then by the third (of thoſe firſt ſet down) multiply all the preceding Numbers, ſetting the Products under the third; and ſo on, by every ſucceeding one (of the Numbers firſt ſet down) multiply all the preceding Numbers, ſetting the Product under their Multiplier.

But obſerve, That when you come to uſe for a Multiplier any ſuperior Power of any of the Primes, you muſt not by it multiply any of the leſſer Powers of the ſame Root, nor any of the Numbers ſtanding under them; only multiply all the other Numbers preceding theſe, *i. e.* all the ſame Numbers which were multiplied by the Root. And to get theſe Products moſt conveniently, take the Root, and by it multiply all the Numbers ſtanding under it, and ſet theſe Products under the Squares; then multiply all theſe Products ſet under the Square, by the ſame Root, and ſet the Products under the Cube, and ſo on.

Exam. 210; its Component prime Factors are 2, 3, 5, 7; for $2\times3\times5\times7=210$. And its ſeveral Meaſures are theſe following, diſpoſed and found according to the Rule.

2	3	5	7
	6	10	14
		15	21
		30	42
			35
			70
			105
			210

Firſt I ſet down 2, 3, 5, 7, then I multiply 2 by 3, and ſet the Product 6 under 3; next I multiply 2, and alſo 3, 6, by 5, and ſet the Products 10, 15, 30 under 5; then I multiply all the preceding Numbers by 7, and ſet the Products under 7; and all theſe Numbers are the Numbers ſought.

Exam. 2d. 6552; its Component prime Factors are theſe, 2, 2, 2, 3, 3, 7, 13, whoſe continual Product is 6552, and its ſeveral Meaſures are theſe following; found thus,

7 :	13 :	3 :	9 :	2 :	4 :	8
	91	21	63	14	28	56
		39	117	26	52	104
		273	819	182	364	728
				6	12	24
				42	84	168
				78	156	312
				546	1092	2184
				18	36	72
				126	252	504
				234	468	936
				1638	3276	6552

The Primes 7, 13, are but once involved, and ſo I ſet them firſt down; 3 is twice involved, and I ſet down 3, 9; then 2 is thrice involved, and I ſet down 2, 4, 8. Then I begin and multiply 7 by 13, and ſet the Product 91 under 13. Next I multiply all the preceding Numbers by 3, and ſet the Products under 3; then I multiply by 9 all the Numbers preceding the Column, over which 3, the Root of 9, ſtands, and which I do, by multiplying all the Numbers ſtanding under 3 by 3; then multiplying all the preceding Numbers by 2, I ſet the Products under 2; and for the following Numbers 4, 8, which are Powers of 2, by them I multiply all the Numbers preceding that Column over which the Root ſtands, *i. e.* all theſe which were multiplied by the Root; and this I do by multiplying all the Numbers ſtanding under 2, by 2, and ſetting the Products under 4; then by the ſame 2 I multiply all the Numbers ſtanding under 4, and ſet the Products under 8.

DEMON.

a :	b :	c :	d
	ab :	ac :	ad
		bc :	bd
		abc :	cd
			abd
			acd
			bcd
			$abcd$

Demon. 1°. Suppose any different Primes, a, b, c, d, &c. their Product $a\,b\,c\,d$, &c. is measurable only by these Primes, or the Products of any two or more of them (*Theor.* X.) And to find all these Measures, we shall first suppose only one prime Number, a, that has no Measure but it self (standing alone in the first Column) but suppose another Prime, b, multiplied into it, then the Product ab has for Measures a, b, and ab (which make the first and second Column.) Again, let another Prime be involved, it's evident that Product abc has for Measures all the Measures of the preceding Product ab, with all these, in which the new Prime can be concerned, which plainly can be no other than c it self with its Products, into all the Measures of ab (which are all the Numbers of the preceding two Columns.) Join another primed d, the Product $a\,b\,c\,d$ has for Measures all the Measures of $a\,b\,c$, together with all these Products in which the new Prime d can be concerned, *i. e.* d it self, and its Products, by all the Measures of $a\,b\,c$ (which are all the Numbers of the preceding Columns) and so it goes on whatever Number of different Primes we suppose; which is all according to the Rule.

2°. If any of the different Primes are oftner than once involved, it's evident that all their Powers, to that one whose Index is the Number of Involutions of the Root, are Measures of the given Number. Then having by the Root multiplied all the Numbers standing already in the Columns preceding it, we have all the Measures of the given Number in which that Root is but once involved; and to have those in which it's twice or thrice involved, [or in which its Square or Cube, &c. are severally concerned, according to the different Powers of it involved in the given Number] it's plain we must multiply these several Powers into all the Numbers preceding the Root (*i. e.* those into which the Root was multiplied) but having done this we must not also multiply any of those Powers into any other of them, nor into the Numbers standing under these others, because those new Products would either contain a greater Power of the same Root, than the given Number contains, and so could not measure it (by *Theorem* X.) or would coincide with the Products of some of the higher Powers, by the Numbers preceding the Root. Thus, in the preceding *Exam.* 2d. if we multiply all the Numbers standing in the Column, which has 2 in the Top, by 4, this will be the same as the Column which has 8 on the Top; all which Numbers under 8 are the Products of 8, by all the Numbers preceding the Root 2, because the Numbers under 2 are the Products of all the preceding Numbers by 2, which Products therefore multiplied again by 4, will be equal to the Products of these preceding by 8, since $4\times2=8$. Again, we are not to multiply any of these in the Columns under 2 or 4 by 8, because the Numbers of the one of these Columns have 2 involved in them, (being the Products of all the preceding Numbers by 2) and the other has 4 involved (being the Product of the same Number by 4) and consequently if these were again multiplied by 8, they would have a greater Power of 2, than the given Number has, in which 8 is the greatest Power of 2. But then to find the Products of all the Numbers preceding the Root by the higher Powers, it's evident we can find them by multiplying gradually all the Numbers under each Power by the Root: For these under the Root are the Products of the preceding by the Root; therefore these Products multiplied again by the Root are the Products of the same preceding Numbers by the Square; and so on.

Coroll. If it's required to find all the *aliquot* Parts of any Number, find all the Measures of it; these (excluding the given Number it self) are the *aliquot* Parts sought.

Theo-

Theorem XII.

If one Number, A, be Incommensurable to each of two or more other Numbers, B, C, D, &c. 'tis also Incommensurable to their Product B, C, D, &c. and reversely.

Demon. Since A is Incommensurable to each of the Numbers B, C, D, &c. therefore A has in its Composition no Prime common with any in the Composition of any of these (*Cor*: 3d. *Theor*.VII.) and consequently none that's common to the Product of any two or more of these; because the Primes of these Products are no other than the Primes of their several Factors (*Cor*. 5. *Theo*. X.) The *Reverse* is plain from the same Principle; for if A is Incommensurable, or has no Prime common with the Product of any two or more of these Numbers B, C, D, &c. then it has no Prime common with any one of these; for if it had, it would also have a Prime common to the Product, since the Product has no other Prime than what belongs to the Factors.

Schol. Tho' this be the direct Demonstration, yet it may be proved very simply after *Euclid*'s Way, thus: If A and B C are Commensurable, then what measures A, one of two Incommensurables, is Incommensurable with the other, B (*Theo*. II.) and because it is Incommensurable with B, yet measures B C, therefore it measures C (*Theo*. VI.) and because it measures also A, hence A, C are not Incommensurable, contrary to Supposition, and therefore A and B C are Incommensurable. For the same Reason A and B C D ($=B\ C\times D$) are Incommensurable; and so on, whatever Number of Factors you suppose to each of which A is Incommensurable.

For the *Reverse*; if any Number measures A, and any one of these others, it will also measure the Product of them all (which is a Multiple of that one) therefore A is not Prime with B C D E, &c. contrary to Supposition.

Corol. If any Number, A, is Incommensurable to another B, it's so also to all the Powers of that other, as B^2, B^3, &c. These Powers being the Products of Numbers to each of which A is Incommensurable; for they are all the same Number B, since $B^2=B\times B$, and $B^3=B\times B\times B$, and so on.

Schol. In forming the contrary to this Theorem, there must be some Limitations, thus,

1°. If A is Commensurable to B, and C, &c. its so also to their Product B C, &c. for it would be so, though it were only Commensurable to one of the Factors.

2°. If A is Commensurable to B C, &c. 'tis so also to one at least of the Factors (for else it were Incommensurable to the Product) but not necessarily so with them all; as here, 4 is Composite to 6, and to $6\times 9=54$, but not to 9; so that as in the first Part, the contrary is larger, or requires fewer Conditions than the Theorem, in the second Part it extends not so far, or draws not so great a Consequence.

Theorem XIII.

If any Numbers, A, B, C, &c. are Incommensurable, each to each of any other Numbers, M, N, O, &c. then the Product of any two or more of the first Set is Incommensurable to the Product of any two or more of the second Set. And the Reverse.

Demon. This follows either from the same Principle as the last, *viz.* the Numbers, A, B, C, &c. having no common Prime with any of the Numbers M, N, O, &c. none of the Products of any two or more of the one, has any common Prime with any one, or the Product of any two or more of the other. Or we may deduce it from the last, *thus*, A is Incommensurable with M, N, O, &c. by Supposition, and therefore it's so with MN, NO, MO, or MNO (*Theor*. XII.) But the same is also true of B and C; consequently each of these last Products considered now as one Number, is Incom-

 mensurable

mensurable to the Products of any two or more of the former, that is, with A B, A C, B C, or A B C. The same Reasoning is equally good, how many Numbers soever there be in each Set.

For the Reverse, *viz.* That if all the Products of each two, or more Numbers, taken out of each Set, are Incommensurable to one another, so are the several Factors of the one Set to those of the other: This is plain from hence, that if we suppose any two of them, as A and M, to be Commensurable, then that Number which measures each of them, will measure any Product in which they are concerned, and so these Products will be Commensurable, contrary to Supposition.

Coroll. If two Numbers, A and B, are Incommensurable, any Power of the one, as A^n, is Incommensurable to any Power of the other, as B^n or B^m; and the Reverse, if A^n, B^m, are Incommensurable, so are A, B.

Exam. 3 and 5 are Incommensurable, and so are 9, and 25 their Squares; also 27, 25, the Cube of 3 and the Square of 5; wherefore if any Fraction, $\frac{A}{B}$, is in its lowest Terms, any of its Powers is so also, as $\frac{A^m}{B^m}$; and the Reverse.

Theorem XIV.

If any Number, A, measures another B, then will any Power of A, as A^n, measure the like Power of the other, B^n. And reversely, If A^n measure B^n, so will A measure B; and also, every other Power of A will measure the like Power of B.

Demon. A has no Prime but what's in B, nor any oftner involved, else it could not measure it; but the prime Factors of A being equally involved in A^n, as those of B are in B^n, it follows, that as there is no Prime in A^n but what is B^n, so there is none oftner involved, and consequently A^n measures B^n. The Reverse is plain from the same Principles. Or we may make the whole Demonstration as simply thus.

$A = 3$	measures	$B = 6$	by 2
$A^2 = 9$		$B^2 = 36$	4
$A^3 = 27$		$B^3 = 216$	8

Let A measure B by D, that is, $B \div A = D$; then is $A : B :: 1 : D$, and (by *Cor.* 11th *Theo.* III. *Book* IV. *Chap.* IV.) $A^n : B^n :: 1 : D^n$; but 1 measures D^n, therefore so does A^n measure B^n. For the Reverse, Let $B^n \div A^n = D$, which is an Integer by Supposition; then $A^n : B^n : 1 :: D$. But the first 3 being Powers of the Order n, therefore (by *Theo.* XII. *Book* IV. *Chap.* IV.) D is a Power of the Order n: Suppose it $= X^n$, so that $A^n, B^n :: 1 : X^n$; wherefore (by *Cor.* 2. *Theo.* III. *Ch.* IV.) $A : B :: 1 : X$. But 1 measures X, and consequently A measures B, and hence every other Power of A will measure the like Power of B.

Theorem XV.

If any Composite Number measures another like Composite, so will the several Factors of the first measure the correspondent Factors of the other.

Thus, if A, B are like Composites, and a, b two of the similar Factors; then if A measure B, so will a measure b; and *Reversely*, If a measure b, so will A measure B.

Demon. Like Composites are in the Ratio of the like Powers of any of the correspondent Factors, that is, $A : B :: a^n : b^n$ (by *Theo.* V. *Book* IV. *Chap.* IV.) But by Supposition A measures B, therefore a^n measures b^n, and therefore (by the last) a measures b.

For the Reverse, If a measures b then does A^n measure B^n (by the last) but also $a^n : b^n :: A : B$, and a^n measures b^n, therefore A measures B.

The-

THEOREM XVI.

If two Numbers, A, B, are composed of an equal Number of different Primes, they cannot be like Composites.

$A=3\times5\times13$, &c.	DEMON. 1°. They cannot be so, by taking these Primes for the similar Factors; because if four Numbers are : : *l* they cannot be
$B=2\times7\times11$, &c.	all Primes (*Cor.* 3d. *Theo.* V.) consequently no two of the prime Factors of the one Compound are : : *l* with any two of the other;

therefore, *Lastly*, Though these Compounds be similar, yet the Similarity does not depend on the Factors being taken in that manner.

2°. They are not Similar by taking any Products of these prime Factors for the similar Factors; for any one, or the Product of any two or more of the Primes of the one Compound, is Incommensurable with any one, or the Product of any two or more of the Primes of the other (*Theo.* XIII.) therefore let us take any two of such Products out of the Primes of each Compound, or a single Prime, and a Product out of each, or two Primes out of the one, and two Products out of the other, or, *Lastly*, a Prime and a Product out of the one, and two Primes or two Products out of the other; yet these cannot be : : *l*, because each Couplet of the compared Terms are Incommensurable, which is inconsistent with Proportionality; for if $a:b::A:B$, then if a, b are Incommensurable, A, B, are Commensurable (by *Cor.* 2d. *Theo.* IV.)

Lastly, Since Numbers can be but one particular Way composed of prime Factors, and no other Numbers can measure them, but these or their Products; and since, as we have now seen, neither these nor their Products can make similar Factors, they can have none such, *i. e.* they cannot be like Composites, or they cannot be resolved into an equal Number of similar Factors.

PROBLEM V.

To find the least common Multiple to any given Numbers, A, B, C, D, &c. which are all different.

N. B. For brevity we shall put ×*le*, for Multiple; *co* : ×*le*, for common Multiple; and *l* : ×*le*, for least common Multiple.

Case I. *For two Numbers*, A, B.

Rule (1^mo^.) If they are Incommensurable, as 4, 7, their Product $AB=28$ is their *l* : ×*le*. But, (2°.) If they are Commensurable, find the two least in their Ratio, as $a:b$ (by *Cor.* 3. *Theor.* III.) so that these are : : *l*, *viz.* $A:B::a:b$, then is $Ab=aB$, the Number sought.

	DEMON. 1°. If A, B are Incommensurable, then is $A\times B$ their
Exam.	*l* : ×*le*. It is a *co* : ×*le*; and to shew that it is the *l* : ×*le*, let
$A=4$. $B=18$	A and B measure any other Number, as N, and let the Quotes
$a=2$. $b=9$	be C, D, thus, A (N=C, and B) N=D; but the Dividend N
$Ab=aB$ $=36$	being common, the Divisors and Quotes are reciprocally : : *l*,
A : B	*that is*, A : B : : D : C; and A, B being Incommensurable, they
AB	measure D, C equally (*Theor.* II.) Again, N=A C and A B :
N	A C : : B : C, therefore A B : N : : B : C. But B measures C,
C : D	therefore A B measures N, which is therefore greater than A B,
	or its equal; consequently any other Number than A B, which

is a *co* : ×*le* to A and B, is greater than A B, therefore this is the *l* : ×*le*.

2°. If A, B are Commensurable, and $a:b$, the least in the same Ratio with A : B, then is Ab, or aB; a *co* : ×*le* to A, B; for Ab is ×*le* of A, and aB of B, also $Ab=aB$. Again, I say, $Ab=aB$ is their *l* : ×*le*, for let them measure any other Number, as N,

 by

by C, D, so that as before, A : B : : D : C; then because A : B : : *a* : *b*, therefore *a* : *b* : : D : C; but *a* : *b* being Incommensurable, *a* measures D, and *b* measures C equally; also, A*b* : A C : : *b* : C, and A C=N, therefore A *b* : N : : *b* : C; and since *b* measures C, so will A *b* measure N, which is therefore either the same Number, or greater then A *b*; hence A *b* is the *l* : ×*le* sought.

Coroll. The least *co* : ×*le* of two Numbers, A, B, measures all their other *co* : ×*les*, which therefore are the Multiples of it; for it's proved that A B, or A *b*, measures any other Number, N, which is supposed to be a *co* : ×*le* to A, B.

Case II. *For more than two Numbers, as* A, B, C, D, &c.

Rule. Find the *l* : ×*le* to any two of them; and then the *l* : ×*le* to the Number last found, and another of them; and so on till you go through them all; and the last found is the Number sought.

Exam.

4 :	6 :	7 :	8
	12 :	84 :	168
A :	B :	C :	D
	m :	n :	o

Demon. Let A, B, C, D, be any Numbers; and the *l* : ×*le* of A, B be *m*; of *m*, C, be *n*; and of *n*, D, be O. I say *n* is the *l* : ×*le* of A, B, C; and O, that of A, B, C, D; for,

1°. A, B measure *m*, and *m*, C measure *n*, therefore A, B, C measure *n*. Again, *n*, D measure O, and A, B, C, measure *n*, therefore A, B, C, D, measure O; and so it proceeds for ever, *i. e.* each Number found in the Course of the Operation is a *co* : ×*le* to all the given Numbers so far.

2°. They are their least *co* : ×*les*; for what is a ×*le* of A, B, is so of *m*, and what is a ×*le* of *m*, C, is so of *n* (*Coroll. Case* I.) therefore what is a ×*le* of A, B, C, is so of *n*, and consequently it is not less than *n*; which is therefore the *l* : ×*le* of A, B, C. Again, what is a ×*le* of A, B, C, is so of *n* (by the last Step) and what is a ×*le* of *n*, D, is so of O (*Coroll. Case* I.) therefore what is a ×*le* of A, B, C, D, is so of O, and consequently is not less than O, which is therefore the *l* : ×*le* of A, B, C, D. The same Reasoning is evidently good from one Step to another for ever; from which we have gained the following Truth, *viz.*

Gen. Coroll. The least common Multiple of any Numbers, A, B, C, &c. is an *aliquot* Part of all their other *co* : ×*les*, or these are Multiples of that.

Scholium.

1*st*. The preceding general Corollary may be demonstrated independently of any Case of this Problem. *Thus*, Take any Number, N, which *l*, the *l* : ×*le* of A, B, C, &c. does not measure, I say it can be none of the *co* : ×*les* of A, B, C, &c. for since *l* does not measure N, (the Quote of *l*) N is a mix'd Number, suppose, $A+\frac{r}{l}$. Hence, by the Nature of Division, N=A*l*+*r*. Now A, B, C, &c. do each by Supposition measure *l*, and consequently they measure its Multiple A*l*; but *l* is the least Number they measure, therefore they cannot measure *r*, which is less than *l*, being the Remainder of a Division in which *l* is the Divisor. *Lastly*, What measures one Part, and not the other Part *r*, cannot measure the whole A*l*+*r*, which being equal to N, consequently A, B, C, &c. cannot each of them measure N, a Number which their *l* : ×*le* *l* does not measure.

2*d*. Though it be true that *m* measures *n*, yet we cannot hence conclude, that the *l* : ×*le* of certain Numbers is a greater Number than that of a Part of these Numbers, because they may happen to be equal; so *m* may be equal to *n*, as in this Example. The *l* : ×*le* of 3, 4 is 12, which is also the *l* : ×*le* of 3, 4, 6. This however is certain, that the *l* : ×*le* of the whole given Numbers can never be less than that of a Part of them; for it's shewn, that *m* must measure *n*; or, for this obvious Reason, that the *l* : ×*le*

$l:\times le$ of the Whole is a $co:\times le$ to any Part of them, and can't be less than their $l:\times le$, which would be absurd.

Again, the Case in which it happens that the $l:\times le$ of the Whole is equal to that of a Part, is this, *viz.* when one of the given Numbers is equal to, or an *aliquot* Part of the least $co:\times le$ of a Part; for then it's manifest from the Manner of the Operation, that this Equality must happen, as you'll see by examining these Examples:

Given Numbers	3 . 4 . 6	3 . 4 . 5 . 12
$l:\times les$	12 . 12	12 . 60 . 60

3*d*. If Numbers are given, to find their $l:\times le$; and if several of these Numbers are the same, or equal, as A, A, B, C, It's plain that we have no more to do, but find the $l:\times le$ to all that are really different Numbers. But if we should apply the Rule to all the given Numbers, without considering that some of them are the same, the same Number must necessarily answer for the $l:\times le$; the Reason of which will also appear from the Nature of the Operation; as here, the $l:\times le$ of 3, 4 is 12; and if to the given Numbers you join another 4, the $l:\times le$ is not thereby changed, because 12 being a $\times le$ of 4, their Ratio in lowest Terms is 3 : 1; so that the Number found upon joining of the new 4, must necessarily be the same as the last, for 12 : 4 : : 3 : 1, and according to the Rule 12×1 is the Number sought, which is the last found $co:\times le$.

4*th*. This Problem is to the same Purpose with this, *viz.* To find the least Number that has Parts denominated by certain given Numbers; for a Number which has a Part of the Denomination A, B, *&c.* must be measurable by A and by B, *&c.* and the least Number which has such Parts must be the least Number measurable by (or the least common Multiple of A, B, *&c.*

Theorem XVII.

The prime Number which measures the $l:\times le$ of certain Numbers, A, B, C, D, *&c.* will also measure some one of these Numbers.

Demon. 1°. Take two Numbers, A, B, if they are Incommensurable, then is AB their $l:\times le$; and if a Prime p measures A B, it measures A or B (*Coroll.* 1. *Theo.* VI.) If A, B are Commensurable, let a, b, be Incommensurable (or least in the same Ratio) then is $Ab=aB$, the $l:\times le$ of A, B; and if p measure Ab and aB, and does not measure A, nor B, it must measure both b and a (*Coroll.* 1. *Theor.* VI.) which therefore are not Incommensurable, contrary to Supposition.

A . B . C . D
m . n . o

2°. If there are more then two Numbers, as A, B, C, D; suppose m the $l:\times le$ of A : B, and n the $l:\times le$ of m : C (*i. e.* of A, B, C.) Again, o the $l:\times le$ of n : D (*i. e.* of A, B, C, D); then if p measures o, it must measure n or D (by the first Article) if not D then n, and consequently it measures either m or C; if not C then n, and consequently it measures either A or B; so that it must necessarily measure one of the Numbers A, B, C, or D; but howmany Numbers soever there be, the same Reasoning will evidently hold through them all.

Theorem XVIII.

Of certain Numbers, A, B, C, D, *&c.* if each of them be separately Incommensurable to any Number, N, so is M their $l:\times le$.

Demon. If any Number measures N, M, then some prime Number measures them (*Theor.* VII. *Coroll.* 3.) which therefore (by the last) measures some one of the given Numbers; consequently each of these is not Incommensurable to N, contrary to Supposition.

Theo-

THEOREM XIX.

The greatest Common Measure, m, and the least Common Multiple, M, of two Numbers, A, B, are reciprocally Proportional with them; thus, $m : A :: B : M$.

DEMON. Take $a : b$, least in the Ratio of A, B, then is $Ab = aB = M$; and $A = ma$, $B = mb$. Hence $Ab = amb = M$: But it's manifest that $m : am :: bm : bam$, that is, $m : A :: B : M$.

PROBLEM VI.

Any Number of Ratios being given, to continue them in Integers, in any Order, and in their least Terms.

Rule. This *Problem* may be solved two different Ways.

Method 1. Continue them by *Problem* I. *B.* IV. *Ch.* IV. and then, reduce the Numbers found to their least Terms (by *Corol.* 3. *Theor.* III.)

Observe, If the given Ratios are in their lowest Terms, then being continued in this Manner, the Numbers found, will, in some Cases, be in the least Terms, but not in every Case, as these Examples shew, *viz.* $3 : 8$, and $4 : 7$, which are in their lowest Terms, being continued, make this Series; $12 : 32 : 56$, which has a Common Measure, 4, which reduces it to this, $3 : 8 : 14$. Again, These, $2 : 3$, and $4 : 7$, make this Series, $8 : 12 : 21$, which is in its lowest Terms. But if the given Ratios are equal, or the same Ratio to be continued, then this Method is good. But I shall refer the Demonstration till after the second general Method is explain'd.

$A : B,\ C : D$
$E : F : G$
$a : b : c$

Method 2. The given Ratios being in their lowest Terms (or reduced to such.) Then suppose only two Ratios, as $A : B$, and $C : D$; they are continued thus; Find F, the *l:×le* of $B : C$, (*Prob.* V.) then find E and G, by these Proportions, *viz.* $B : A :: F : E$, and $C : D :: F : G$; then are $E : F : G$ the Numbers sought.

DEMON. That $E : F : G$, continue the given Ratios, is manifest from the Construction: For $A : B :: E : F$, and $C : D :: F : G$. And that they are the least Numbers which continue the given Ratios, is thus proved: Suppose any other Numbers, $a : b : c$, which are in the given Ratios; I say, these are greater Numbers than $E : F : G$; for, $A : B :: a : b$; but $A : B$ are least in their Ratio; therefore B measures b. Again, $C : D :: b : c$; but $C : D$ are least in their Ratio; therefore C measures b; therefore also, F, the *l:×le* of B, C, must measure b (by *Gen. Corol. Prob.* V.) Again, Since $F : b :: E : a :: G : c$, and F measures b; therefore E and G, do equally measure a and c; consequently a, b, c, are greater than E, F, G, which are therefore the least that continue the given Ratios.

$A : B,\ C : D,\ E : F$
$G : H : I$
$K : L : M : N$
$a : b : c : d$

II. Suppose 3 given Ratios, $A : B$, $C : D$, $E : F$. Find as before, $G : H : I$, the 3 least Numbers that continue the 2 first Ratios; then find M, the *l:×le* to $I : E$. And lastly, find K, L, N, by these Proportions, $I : M :: H : L :: G : K$, and this $E : M :: F : N$; then are these the Numbers sought, *viz.* $K : L : M : N$.

DEMON. 'Tis plain they continue the given Ratios; and they are the least that do so: For suppose any other Numbers, that also continue the given Ratios, as $a : b : c : d$. I prove that they must be greater than $K : L : M : N$; thus, $A : B :: a : b$; but $A : B$ are least in their Ratio; therefore B measures b. Again, $C : D :: b : c$, and $C : D$ are least in their Ratio; therefore C measures b. Hence again, H, the *l:×le* of B, C, measures b; but also $H : I :: b : c$; therefore I measures c. Again, $E : F :: c : d$,

$c:d$, and $E:F$, being in least Terms, E measures c; therefore, M, the $l:\times le$ of $I:E$, measures c, but $M:c::K:a::L:b::N:d$; and because M measures c, therefore K, L, N, do equally measure a, b, d: Hence a, b, c, d, are greater than K, L, M, N, which are therefore the least that continue the given Ratios.

III. Suppose four Ratios, $A:B$, $C:D$, $E:F$, $G:H$; continue the first two by the Numbers, $I:K:L$ (K being the $l:\times le$ of B, C) and continue the first three in the Numbers $M:N:O:P$, (O being the $l:\times le$ of $L:E$) then find T, the $l:\times le$ of $P:G$. And lastly, find Q, R, S, U, by these Proportions, *viz.* $P:T::M:Q::N:R::O:S$, also $G:T::H:U$; then Q, R, S, T, U, are the Numbers sought.

$A:B,\ C:D,\ E:F,\ G:H$
$I:K:L$
$M:N:O:P$
$Q:R:S:T:U$
$a:b:c:d:e$

DEMON. They continue the given Ratios plainly; and that they are the least Numbers which do so, I prove, by shewing that any other Numbers, a, b, c, d, e, which continue the same Ratios, are greater than they: Thus, $A:B::a:b$, and $C:D::b:c$; therefore B and C do both measure b; and consequently, K, the $l:\times le$ of B, C, does measure b; and because $K:L::b:c$, therefore L measures c. Again, $E:F::c:d$, therefore E measures c: And hence again, O, the $l:\times le$ of L, E, does also measure c; and because $O:P::c:d$, and $G:H::d:e$; therefore both P and G measure d; and consequently T, the $l:\times le$ of P, G, does measure d: But now $T:d::Q:a::R:b::S:c::U:e$; and because T measures d, therefore Q, R, S, U, do equally measure a, b, c, e. Hence a, b, c, d, e, are greater than Q, R, S, T, U, which therefore, lastly, are the least that continue the given Ratios.

As I think the Progress of this Rule, and its Reason, *ad Infinitum*, will be clearly perceived after what's explained, I shall carry it no further; but only make this general *Remark*, *viz.* That for every succeeding Series, or every new Ratio added, we begin always with finding the $l:\times le$ of the Antecedent of the new Ratio, and the last Term of the preceding Series; and making that the last but one of the Series sought, we find the rest by Proportions drawn from this and the Terms of the preceding Series, together with the Ratio added; then the Term which was first found, is a principal *Medium* for demonstrating that the Numbers found continue the given Ratios in the least Terms.

SCHOLIUMS.

1*st.* Either of these two Methods are *universal*, whether the given Ratios are different or the same; and are indeed the only Methods that solve this *Problem*, in all Cases; but in that Case where the same Ratio is to be continued, if we take it in its least Terms, and continue it by the *Problem* refered to in the first *Method*, the Series found will be in its least Terms; as I have already said in an Observation after the first *Method*; and which I shall now demonstrate: Thus,

$$\begin{array}{l} 2:3 \\ \quad 2:3 \\ \hline 4:6:9 \\ \qquad 2:3 \\ \hline 8:12:18:27 \\ \qquad\quad 2:3 \\ \hline 16:24:36:54:81 \\ \quad \&c. \end{array}$$

By the Method of this Operation it's manifest, that the Extremes of the Series are like Powers of the Terms of the given Ratio; and these being Incommensurable (or in their least Terms) their like Powers must be so (*Theor.* XIII. *Cor.*) But the Extremes of a Series being Incommensurable, the Whole must be so.

Or, The Truth of this will also appear from the preceding second *Method*: For if you compare this Operation exactly with the preceding *Rule*, you will find it's the very same Work which would be made by that *Rule*; only in this particular Case it is easier Work, because of its being the same Ratio, and in its least Terms.

Ob-

Observe also, That the same Series is made by the Series of the Powers of the given Terms of the Ratio, multiplied together in a reverse Order; as has been explained in *Schol.* to *Theor.* VIII. *B.* IV. *Ch.* IV. and which is expressed in this general Form:

$$A^n : A^{n-1} \times B : A^{n-2} \times B^2 : \&c. : A \times B^{n-1} : B^n.$$

2*d.* In *B.* IV. *Ch.* IV. *Prob.* I. *Schol.* 2*d.* I have observed, That tho' we take all the possible Expressions of the same Ratio, and continue a Series by each of them, yet this would not exhaust all the Variety of Numbers, in which a Series in the same Ratio might appear; and the universal Method of finding all that Variety was referred to another Place: And here we plainly have it: Thus,

Raise a Series from the least Terms of the given Ratio, to the proposed Number of Terms; then successively multiply this Series by 2, 3, 4, *&c.* taking the Multipliers in the Order of the natural Series, *ad Infinitum*; and you shall have hereby the Series in all its possible Variety: For it is first in its lowest Terms, and then in all the Multiples of these; which necessarily exhaust all the Variety: Because a Series is either in its least Terms, or in the Multiples of these (*Theor.* I.) Therefore, if A : B express the least Terms of any Ratio, all the possible Variety may be expressed in this general Form:

$$A^n x : A^{n-1}Bx : A^{n-2}B^2x : \&c. : AB^{n-1}x : B^n x.$$

Where, according to the infinite Variety of Numbers, in the Order of the natural Series, that x may represent, so will the Series be different: So x, being 1, the Series is in its least Terms; because 1 does not multiply; but in all other Values of x, the Series is Multiple of the least Terms.

Observe also, That if x is any Number, which is a Power of the Order n ($n+1$, being the Number of Terms of the Series) then the Series, with that Value of x, is what would arise by working, according to the Rule in *Schol.* I. with some of the greater Terms of the Ratio, which are always Equimultiples of the least Terms; *Thus*, Let $x = r^n$, then is $A^n x = A^n r^n = \overline{Ar}|^n$ (by *Theor.* I. *B.* III. *Ch.* I.) and $B^n x = B^n r^n = \overline{Br}|^n$, and these Extremes are plainly like Powers of $Ar : Br$, Equimultiples of the least Terms of the Ratio A : B. But all the greater Terms of this Ratio are universally expressed by $Ar : Br$; and the Extremes of a Series rais'd from these, are universally $\overline{Ar}|^n : \overline{Br}|^n$: Which shews that all the Variety that would arise by working with all the different greater Terms of the Ratio, is had by making x any Power of the Order n. And lastly, If x is any Number, which is not a Power of the Order n, then we have all the rest of the Variety, which cannot arise from working according to the preceding Rule; because whatever Terms of the Ratio we work with, as $Ar : Br$, the Extremes will be $\overline{Ar}|^n : \overline{Br}|^n$, similar Powers; but x not being a Power of the Order n, $A^n x : B^n x$ cannot be such Powers (*Cor.* 4. *Theor.* II. *B.* III. *Ch.* I.)

Observe, lastly, That in a Multiple Ratio, or whose lesser Extreme is 1, all the Variety is had by raising a Series from all the different Expressions of the Ratio; which in this Case only has the same Effect, as multiplying the least Terms of the Series.

§. II. *Relating all to* Geometrical Progressions.

Theorem XX.

If a Geometrical Series is in its least Terms (*i. e.* all the Terms Incommensurable) the Extremes are such like Powers of the least Terms of the Ratio, whose Index is the Number of Terms less 1. And if the Series is not in its least Terms, the Extremes are Equimultiples of these like Powers of the least Terms of the Ratio.

Demon. This is plain from the Method of raising a Series in its least Terms, explained in the preceding Problem. (See *Schol.* 1.) For if a Series rais'd from the least Terms of the Ratio, is in its least Terms, and the Extremes are the $n-1$ Powers of the least Terms of the Ratio; also, since two Series in the same Ratio cannot be both in least Terms, and consist of different Numbers; therefore the first Thing propos'd is manifest. Again, if the Series is not in least Terms, it consists of Equimultiples of the least Terms, (see *Schol.* 2. preceding *Problem*) and consequently the Extremes are Equimultiples of the $n-1$, Powers of the least Terms of the common Ratio.

Schol. As every Series, not in its least Terms, are Equimultiples of the least; so, according as the common Multiplier is, or is not, a Power of the Order n, the Extremes will be, or will not be, such Powers: And comparing this with the Extremes of the Series, in its least Terms, we may observe, that the Extremes of every Series have one of these three Qualities.

1°. They are Powers of the Order n; but not also Equimultiples of other Numbers, which are such Powers: And this happens only when the Series is in least Terms.

2°. They may be Powers of the Order n, and also Equimultiples of such Powers; which happens only when the least Terms of the Series are multiplied by some Power of the Order n; or, when a Series is raised from any such Terms of the Ratio as are not the least.

3°. They are Equimultiples of Powers of the Order n; but are not such Powers themselves: Which happens when the least Terms of the Ratio (or Series) are multiplied by some Number which is not a Power of the Order n.

COROLLARIES.

1*st*. If four Numbers are : : *l*, A : B : : C : D; and if any two of the Comparative Terms, as A, B, or A, C, are Similar Powers of any Order; the other two, C, D, or B, D, are either Similar Powers of the same Order, or they are Equimultiples of such Powers: For if A, B are Powers of the Order n, they admit $n-1$ Means; and so also do C : D; consequently, by this *Theorem*, they are either Powers of the Order n, or Equimultiples of such.

2*d*. Two Numbers, A, B, that are like Composites of n Factors, are either both Powers of the Order n, or Equimultiples of such; for being like Composites of n Factors, they admit of $n-1$ Means; *i. e.* they are the Extremes of a Geometrical Series of $n+1$ Terms; (*Theor*. IX. *B*. IV. *Ch*. IV.) and consequently, by this *Theorem*, they are either, &c.

Theorem XXI.

If a Geometrical Series is in its least Terms, or Incommensurable, so also are the Extremes.

 Demon.

DEMON. The Extremes are like Powers of the least Terms of the Ratio; but the least Terms are *Incommensurable* (*Theor*. IV.) and their like Powers are also *Incommensurable* (*Theor*. XIII. *Coroll*.)

COROLL. If the Extremes of a Series are *Commensurable*, the whole Terms are so; for the Whole being *Incommensurable*, so also are the Extremes.

THEOREM XXII.

In every Geometrical Series, whose Ratio is not Multiple, the whole Terms, excluding either of the Extremes, are *Commensurable*.

DEMON. By the Work of the preceding *Problem*, as it is represented in *Schol*. 1. all the Terms of the Series are Multiples of the lesser Term of the Ratio, except the greater Extreme; and all are Multiples of the greater Term of the Ratio, except the lesser Extreme: Wherefore in all Cases, these Terms are Commensurable by that Term of the Ratio.

COROLLARIES.

1*st*. All the Terms of every Geometrical Series, except the lesser Extreme, in some Cases, are Composite Numbers: For if the Ratio is Multiple, the lesser Extreme may be a Prime, and then all the other Terms are Multiples of it: But if the Ratio is not Multiple, all the Terms are Composite Numbers; which are either Powers or Multiples of the Terms of the Ratio. Hence again,

2*d*. No Term of any Geometrical Series, except the lesser Extreme, can be a Prime Number. And hence again,

3*d*. Betwixt two Prime Numbers there cannot be a Geometrical Mean in a whole Number, nor consequently in a mix'd Number; because the Mean is the square Root of the Product of the Extremes; which not having an Integral Root, has none at all (by *Theor*. XIX. *B*. III. *Ch*. I.) And more generally, betwixt two Primes there falls no Number of Geometrical Means; for Integral Means they cannot be, by this *Theorem*; and they cannot be Fractional, as you'll see in *Theor*. XXV.

THEOREM XXIII.

If a Geometrical Series, whose Ratio is not Multiple, is in its least Terms or *Incommensurable*, another Integral Term cannot be added, either increasing or decreasing.

DEMON. The Series being in its least Terms, if we suppose another Integral Term added, then in this increased Series, the whole Terms, excluding this new Extreme added, are not Commensurable; which is contrary to the last *Theorem*.

Or take this other *Demonstration*: Let A, B and L, be the first, second and last Term of a Series, in its least Terms; to which let another Term, x, be added; then is A : B : : L : x. But A : L are *Incommensurable*; because the Series from A to L is so (*Theor*. XXI.) therefore A measures B. And hence again, A measures every Term of the Series; the Ratio being in this Case Multiple, *i. e.* every Term being an *aliquot* Part of the next greater, and consequently of every greater: But if A measures L, then A, L are Commensurable, contrary to Supposition.

COROLL. To two Numbers *Incommensurable*, a third in Geometrical Proportion, cannot be an Integer.

SCHOLIUMS.

1°. The Sense of this *Theorem* is the same with this, *viz*. If a Series is in its least Terms (the Ratio not Multiple) the last Term cannot be to any Integral Number in the same Ratio, as the first Term to the second.

2°. Though the Extremes of a Series are Commensurable, and at the same Time also both Composite Numbers; yet it will not follow that another Integer can be added to the Series; as here, 20 : 30 : 45; the Extremes, 20, 45, are both Composite Numbers,

Numbers, and Commensurable; yet another Term in the same Ratio is not Integral; for it is $47\frac{1}{5}$. But this Truth we may also demonstrate independently of any Particular *Example*. Thus, In every Series, all the Terms, except the lesser Extreme, are both Commensurable and Composite Numbers: And if the Series is in its least Terms, another Integral Term cannot be added; but if we take the Term next the lesser Extreme, and the greatest Extreme, for the Extremes of a Series, they are both *Commensurable* and *Composite*; yet another Integral Term cannot be added above the greater Extreme; because what is added to this Series, is also added to the Series of which it is a Part, and to which another Integral Term is shewn to be impossible; because its Extremes are Incommensurable.

3°. As to the special Character of a Series, which admits of another Integral Term, it is to be deduced from *Theor.* X. *Thus*, Let A, B, L, be the first, second, and last Terms of a Geometrical Series; if another Term added after L is Integral, let it be called M; then is A : B : : L : M; so that A measures B×L; (for $M = BL \div A$.) Consequently there is no Prime in the Composition of A, but what is found in the Composition of BL (*i. e.* either in B or L) nor any Prime oftner involved in A, than it is in BL: For otherwise A could not measure BL (by *Theor.* X.) and so M would not be Integer.

Thus then we see what are the Conditions of three Integral Numbers, that admit a fourth Proportional, which is also an Integer.

THEOREM XXIV.

If there be any one Choice of two Terms in a Series, whereof the lesser measures the greater, then every lesser shall measure every greater: And if there be any two Terms, whereof the lesser does not measure the greater, then none of the lesser shall measure any of the greater.

DEMON. 1*st*. Let any Series be thus represented, $A : Ar : : Ar^2 : Ar^3$, &c. If any Term is divided by any lesser, the Quote is some Power of r; this is manifest; and if the Quote is an Integer, then r is an Integer; because any Power being Integer, its Root must be so too (*Theor.* XIX. *B.* III. *Ch.* I.) Again, If r is an Integer, all its Powers are Integers; wherefore every lesser Term measures every greater.

2*d*. The second Part is obvious from the preceding; for if any lesser Term measures any greater, all the lesser would measure all the greater, contrary to Supposition.

COROLL. If A, the lesser Extreme of a Series, measures the second Term, B, it is the greatest common Measure of the whole Series; for it measures all the other Terms, and is its own greatest Measure.

THEOREM XXV.

If A, the lesser Extreme of a Series, is a Prime Number, the Ratio of that Series is Multiple; or every lesser Term measures every greater.

DEMON. If A does not measure the second Term, B, then being a Prime Number, A, B, are *Incommensurable*; and consequently there cannot be a third Integral Term (by *Coroll. Theor.* XXIII.) contrary to Supposition; and if A measures B, then every lesser Term measures every greater.

COROLL. If the lesser Extreme of a Series is a Prime Number, that Series cannot be in its least Terms; because the lesser Extreme measures the Whole.

THEOREM XXVI.

Whatever Number measures the Extremes of a Series, will measure all the middle Terms; or thus, the common Measure of the Extremes is so to the whole Series.

DEMON. Let the Series be, $A : B : C$, &c. $: : L$, and suppose that m measures the Extremes A, L, by these Quotes, a, l; that is, $\frac{A}{m}=a$, and $\frac{L}{m}=l$; then is $A : L : : a : l$; and as many Means as fall betwixt $A : L$, so many fall betwixt $a : l$ in the same Ratio. Let the second Series be a, b, c, &c. l; then is $A : a : : B : b$; but $A=ma$, therefore $B=mb$, hence $\frac{B}{m}=b$; *i. e.* m measures B by b. In the same manner will the Reasoning proceed to the next middle Term C; for, $B : b : : C : c$; but $B=mb$, and therefore $C=mc$, and $\frac{C}{m}=c$; and so of all the rest.

COROLL. Hence we have an easy Rule for finding the greatest common Measure of any Series, *viz.* by finding that of the Extremes.

SCHOL. This Theorem is true, whether the Series be all Integers or not, and whether m be so or not.

THEOREM XXVII.

As many Geometrical Means as fall betwixt any two Numbers, A, L, so many there fall betwixt each of them, and their greatest common Measure.

DEMON. 1°. If A, L are the least in their Ratio, then whatever Number of Means is supposed to fall betwixt them, as $n-1$, the whole Series is least in its Ratio, and A, L are Powers of the Order n (*Theor.* XX.) As suppose $A=a^n$, and $L=b^n$; now a^n, b^n being least in the Ratio, 1 is their greatest common Measure, and betwixt $1 : a^n$, also betwixt $1 : b^n$ there fall $n-1$ Means (*Cor.* III. *Probl.* III. *B.* IV. *Ch.* III.)

2°. If A, L are Commensurable, let m be their greatest common Measure, and $A\div m=B$; $L\div m=D$; then are $B : D$ least in the Ratio of $A : L$ (*Theor.* III.) and being in the same Ratio, therefore they admit as many Means (*Theor.* VII. *Book* IV. *Chap.* IV.) But by the last Article, betwixt 1 and B, or D, there fall as many Means as betwixt $B : D$, or $A : L$, as suppose $n-1$. But again, since $A\div m=B$, therefore $m : A : : 1 : B$, and betwixt $1 : B$, there fall $n-1$ Means; consequently there fall as many betwixt $m : A$. And because $L\div m=D$, hence $m : L : : 1 : D$; and betwixt $1 : D$ there fall $n-1$ Means, consequently as many betwixt $m : L$, that is, as many as betwixt $A : L$.

SCHOL. By the same Reason there will fall as many Means betwixt A or L, and any of their common Measures, as fall betwixt $A : L$ themselves.

THEOREM XXVIII.

As many Means as fall betwixt any two Numbers, A, B, so many fall betwixt each of them and their least common Multiple M.

DEMON. Let m be the greatest common Measure of A, B; then (by *Theor.* XIX.) $m : A : : B : M$; and (by *Theor.* XXVII.) there fall as many Means betwixt $m : A$ (or $m : B$) as betwixt $A : B$; also (by *Theor.* VII. *Book* IV. *Chap.* IV.) as many betwixt $B : M$ or $A : M$ as betwixt $m : A$ or $m : B$; *that is*, as many as betwixt $A : B$.

THEOREM XXIX.

Of a Series in continued Proportion, take the Series of the greatest common Measures, or least common Multiples, to every two adjacent Terms; these are also in one continued Proportion.

DEMON.

$A : B : C : D :$ &c.
$l \quad m \quad n$
$a : b$

Demon. 1°. For the greatest common Measures: Let l be that of A, B, and m of B, C, and n of C, D; also, let $a : b$ be the least Terms of the common Ratio of the first Series. Now, l, m, n do equally measure A, B, C, *viz.* by a; and they also measure B, C, D equally by b; (*Theor.* III.) and, *Reversely*, a measures A, B, C by l, m, n, and b measures B, C, D by l, m, n; hence l, m, n are continuedly in the same Ratio as A, B, C; that is, as $a : b$.

2°. For the least common Multiples: Let l, m, n be the l : $\times$*les*, then is $l = Ab$, $m = Bb$, $n = Cb$; hence $l : m : : Ab : Bb : : A : B$, and $m : n : : Bb : Cb : : B : C$; therefore $l : m : : m : n$.

Theorem XXX.

If any Numbers, $\div l$, are in their lowest Terms, as A, B, C, D, and L, the least Terms of whose Ratio are $a : b$; whatever Number, m, measures any Term of the Series, it's Commensurable with a or b.

Demon. The Series being in its lowest Terms, and $a : b$ the lowest Terms of the common Ratio, then the Extremes are $A = a^n$ and $L = b^n$; and any middle Term may be expressed $a^{n-r} \times b^r$. (See *Schol.* I. *Probl.* VI.) Now, if m is Incommensurable with a, and b; it's so with any Power of a and b, and with any Product of any Power of the one by any Power of the other, (*Theor.* XII. *Coroll.*) wherefore it cannot measure any Term of the Series, contrary to Supposition.

Theorem XXXI.

If $A : B : C : D$, &c. are $\div l$, and in their lowest Terms, each of them is Incommensurable with the Sum of all the rest.

Demon. Take any one of them, as, B; I say B, and $\overline{A+C+D}$ are Incommensurable; for if they are Commensurable let m measure both; and take $a : b$, the lowest Terms of the common Ratio; then, since m measures B, it is Commensurable to a or b (by the last); suppose to a; and let n measure m, and a, therefore n measures a, B, and $\overline{A+C+D}$ (because n measures m, and m measures B and $\overline{A+C+D}$). But a, the Antecedent of the lowest Terms of the Ratio, measures all the Antecedents of the Series, A, B, C, &c. (these being all Multiples of A, as appears from the Work of *Probl.* VI. as it is in *Schol.* 1*st.*) And since n measures a, by Supposition, therefore it also measures each of these, A, B, C; but it measures also $\overline{A+C+D}$, therefore it measures D (*Axiom* 3.) consequently it measures each of these, A, B, C, D, which therefore are not least in the Ratio, contrary to Supposition.

If instead of a, we take b, the Consequent of the Ratio, the Demonstration will be the same; for then b measures all the Consequents, B, C, D, &c. and consequently n measures them all; and because it measures $\overline{A+C+D}$, therefore it also measures A, *i. e.* it measures each of these, A, B, C, D, &c. contrary to Supposition.

How many Numbers soever you suppose, and which soever of them you take, the Demonstration will still be the same, from a measuring all the Antecedents, and b all the Consequents.

Theorem XXXII.

Of a Series $\div l$, and in their least Terms, any one of them is Incommensurable to the Sum of the whole Series.

Demon.

DEMON. Any one of them is Incommensurable to the Sum of all the rest (by the last) and this Sum added to that one (which makes the Sum of the Whole) is Incommensurable to any of the Parts added (*Cor. Ax.* 2d.) *viz.* to that one.

THEOREM XXXIII.

If $a : b : c : d$: &c. $k : l$, $\div l$, and in least Terms, do equally measure A : B : C D, &c. K : L, by m; also if $r : s$ are the least Terms of the Ratio, and neither of them does measure m, then another integral Term cannot be added to the last Series, A : B : C, &c. : K : L.

DEMON. Since $a : b : c$, &c. do measure A : B : C, &c. by m, therefore the last Series is the same as $am : bm : cm$, &c. $km : lm$, and the first Series the same as $r^n : r^{n-1}s$, &c. to S^n; then r, s being Incommensurable, so are r, s^n, or $r : l$; and $s : r^n$ or $s : a$. (*Cor. Th.* XIII.) Again, $km : lm :: lm : \frac{l^2m^2}{km} = \frac{l^2m}{k}$, the Term added; which is not Integral; for $\frac{l}{k} = \frac{s}{r}$ hence $\frac{l^2m}{k} = \frac{slm}{r}$, but r is Incommensurable to both s and l, and consequently to sl; (*Th.* XII.) and also it does not measure m, therefore it does not measure slm; (*Theor.* VI. *Cor.* 2d) that is, the Term added is not Integral.

SCHOL. By the same Method of Demonstration it will appear, that if any two Numbers, $a : b$ are least in their Ratio, and do equally measure other two, A, B, by M, which neither a or b measures, then a third $\div l$ to A : B, is impossible in Integers; for A$=am$ and B$=bm$, and the third $\div l$ is $\frac{b^2m^2}{am} = \frac{b^2m}{a}$. But a being Incommensurable to b, is so to b^2, and it does not measure m, consequently does not measure b^2m.

THEOREM XXXIV.

The Extremes of every Geometrical Series, whose Ratio is not Multiple, are like Composite Numbers, whose Index is the Number of Terms less 1: So if the Series has three Terms, the Extremes are composed of two like Factors; if the Series has four Terms, the Extremes are composed of three Factors.

DEMON. The Extremes of every Series are either like Powers, whose Index is the Number of Terms less 1, or they are Equimultiples of such like Powers (*Theor.* XX.) If they are Powers, the Truth of this Theorem is manifest; for like Powers are comprehended under the Notion of like Composites: If they are Equimultiples of like Powers, as $xa^n : xb^n$, the same Truth is also manifest; for they may be resolved into $xa \times a \times a$, &c. and $xb \times b \times b$, &c. repeating a in the one, and b in the other equally; then it's plain that $xa : xb :: a : b$; the rest are all a and b.

But now *observe*, That the Similarity of the Composition may be in some Cases, after a Manner different from any of these already represented; in which all the Factors in each Extreme are equal among themselves, as when they are like Powers; or all except one, as in the other Case; for in some Cases, I say, the Similar Composition will be by Factors, which are all different among themselves in each Extreme: And in some Cases, though they are not all different, yet neither will they be all equal, except one. Now, because the preceding Demonstration represents the Extremes of a Series, as composed after two particular Ways, though there are also others; therefore I shall give another Demonstration of the *Theorem*, unlimited to any particular kind of Similiar Composition, and which comprehends them all: And afterwards I shall explain the Limitations of some particular Cases.

Let the Extremes of a Series be A, N, and the Term next the greater Extreme (N) be M. Then, I say,

1st. If

1*st*. If the *Theorem* be true in every Case where the Number of Terms is n, it is also true when the Number of Terms is $n+1$; which I thus prove:

1°. If from A to N, including both, there are $n+1$ Terms, then from A to N, excluding one of them, there are n Terms; and by Supposition the *Theorem* is true, of n Terms from A to M: Suppose as many Terms, least in their Proportion, as the Series A ... M; the Extremes of this new Series, are, by Supposition, like Composites; which we may represent thus, *viz.* ab&c. cd&c. supposing as many Factors as $\overline{n-1}$. Dispose these under the other, as in the Margin. Then,

A ..... M : N
ab&c. ... cd&c. :

2°. Since ab&c. : cd&c. are Similar Composites, it is $a:c::b:d$; and so through all the Factors, comparing them in Order, the least Factor of the one to the least of the other; and so gradually on to the greatest. Again, this new Series being in the least Terms, and in the same Ratio with the other Series, from A to M, every Term in that Series will measure its Correspondent in this equally, and that by their greatest common Measure, suppose m: Therefore $m\times ab$&c.$=$A, and $m\times cd$&c.$=$M.

3°. Compare the second Series from cd&c. to as many of the other, taken from N, towards the Left-hand, and they are also least in the same Ratio with these; because these are Part of the same Series proceeding from A; consequently each Term of the Series, ab&c. .... cd&c. measures its Correspondent, in this Part of the first Series, equally, suppose by n; thus, cd&c. measures N by n; hence, $n\times cd$&c.$=$N; so that, for the first and two last Terms of the given Series, we have new Expressions equal to them, which are these standing under them, in the annex'd Scheme in the Margin. Then,

A .... M : N
$m\times ab$&c. ... $m\times cd$&c. : $n\times cd$&c.

4°. Since ab&c. cd&c. or A, M, are Similar Composites, by Supposition, it remains only to be shewn that $m\times ab$&c. and $n\times cd$&c. that is, A and N are also Similar Composites; which is done thus, The Series from $m\times ab$&c. to $n\times cd$&c. is in the continued Ratio of m to n; for the two last Terms, *viz.* $m\times cd$&c. $n\times cd$&c. are plainly so; wherefore the second Term of the given Series being called B, its equivalent Expression in the other Form, is $n\times ab$&c.; but $m\times ab$&c. : $n\times cd$&c. :: $\overline{m\times ab}^{n}$&c. $\overline{n\times ab}^{n}$&c. ($=$B^{n}.) the Index n being the Number of Terms less 1. (*Theor.* VI. *B.* IV. *Ch.* IV.) Again, because $m\times ab$&c. : $n\times ab$&c. ($=$B) :: $m:n$; therefore $\overline{m\times ab}^{n}$&c. : B^{n} :: $m^{n}:n^{n}$; (*Cor.* 11. *Theor.* III. *B.* IV. *Ch.* IV.) wherefore also $m\times ab$&c. $n\times cd$&c. :: $m^{n}:n^{n}$. Hence again, ab&c. : cd&c. :: $m^{n-1}:n^{n-1}$ ($\overline{n-1}$ being the Number of Factors in ab&c.) but by Supposition, $a:c::b:d$; so that $\frac{a}{c}=\frac{b}{d}$; and the like being true in the rest of the Factors of ab&c. cd&c. compared, therefore $\frac{a}{c}\times\frac{b}{d}\times$&c.$=\frac{ab\&c.}{cd\&c.}$ $=\frac{a^{n-1}}{c^{n-1}}$; wherefore ab&c. : cd&c. :: $a^{n-1}:c^{n-1}$, and ab&c. : cd&c. :: $m^{n-1}:n^{n-1}$. Hence $a^{n-1}:c^{n-1}::m^{n-1}:n^{n-1}$, and $a:c::m:n$ (*Cor.* 11. *Theor.* III. *B.* IV. *Ch.* IV.) Wherefore, lastly, mab&c. ncd&c. that is, A, N are Similar Composites of $\overline{n-1}$ Factors, n being the Number of Terms in the Series; which is the first Part of the Demonstration. But,

2*d*. The Proposition is true of any three Terms, A : B : C in a continued Series, *i. e.* A and C, are like Composites of two Factors; which is thus demonstrated;

Take

$A : B : C$
$a : b$
$c : d$
$ac : cb : bd$

Take $a : b$, the least in the same Ratio, with $A : B$ and $B : C$; they must equally measure A, B, suppose by c; and B, C, suppose by d; that is, $\frac{A}{a}=c$, and $\frac{B}{b}=c$: Also, $\frac{B}{a}=d$, and $\frac{C}{b}=d$; therefore $ac=A$; $cb=B=ad$, and $bd=C$: Wherefore, $ac : bc : bd$, are in the same Ratio continuedly, since they are equal to $A : B : C$; but it's plain, that $ac : bc :: a : b$, and $bc : bd :: c : d$. Hence $a : b :: c : d$, that is, ac and bd are like Composites of two Factors.

3*d*. Therefore the *Theorem* is true in all Cases; for it is true in Case of three Terms, by *Article* 2d; and it follows from the 1st *Article*, that it's true of four Terms; and from this again it's true of five Terms, and so on for ever.

COROLLARIES.

1*st*. If the Product of two Numbers, A, B, makes a Square Number, these Numbers are like Composites of two Factors; for if $AB=x^2$, then $A : x : B$ is a continued Series, whose Extremes, A, B, are, by this *Theorem*, like Composites of two Factors.

2*d*. If it is $A : B :: C : D$, and if $A : B$ are like Composites of any Number of Factors, C, D are like Composites of the same Number of Factors; because they admit as many Means as A, B, or are the Extremes of an equal Series, with that of which A, B are the Extremes.

SCHOLIUMS.

1*st*. The Reason of limiting the Series to a Ratio, which is not Multiple, is, because if the Ratio were Multiple, the lesser Extreme may be 1, or a prime Number: But as 1 is allowed to be a Power of all Orders, and consequently to be a Composite of 1, as a continual Factor, for it is $1\times1\times1$, &c. If we also allow 1 to be a Factor in other Cases; then the *Theorem* may be taken without the Limitation.

2*d*. It has been shewn, that the Extremes of every Series are like Powers, or the Equimultiples of like Powers; and in some Cases, that they are both. Those that are like Powers only, and not also Equimultiples of like Powers, have a Similar Composition, by equal Factors only. Those that are not like Powers, have a Similar Composition only by Factors that are not all equal; and which, in some Cases, will be all different; in others not. *Lastly*, Those that are both like Powers and Equimultiples of like Powers, have a Similar Composition, both by equal Factors, and by such as are not all equal. I shall explain these Things a little more particularly.

(1°.) Those that are like Powers only, have a Similar Composition only by equal Factors: The Reason of which, is this; Such a Series is necessarily in its least Terms; for if it is not, the Extremes are either Equimultiples of like Powers, or they are so, and also like Powers; and therefore not like Powers only, contrary to Supposition.

Now, if a Series is in its least Terms, the Extremes are like Powers of the least Terms of the Ratio; the Index being the Number of Terms less 1 (*Theor.* XX.) Thus the least Terms of the Ratio being $a : b$, the Extremes are a^n, b^n; but $a : b$ being Incommensurable, so are a^n, b^n (*Coroll. Theor.* XIII.) so that there is no Prime common to a^n and b^n; for which Reason they can never be resolved into any Number of Similar Factors; because these Factors could be no other than the Primes that compose them, or Products made of these Primes, which cannot be Proportional: For suppose that x, z, are any two Products, made of the Primes of a^n; and y, v, two made of the Primes of b^n; these Products are Incommensurable each to each, *i. e.* x to y, and z to v; because the Primes of a^n are all different from those of b^n; wherefore x, y, z, v, are not Proportional; for if they are, then x, y, being Incommensurable, z, v, must be Commensurable, contrary to what's already shewn.

Hence we know how to find whether the Extremes of a Series are similary Composite, by equal Factors only, whose Number is the Number of Terms less 1, *viz.* by finding whether they are in their least Terms.

(2°.) Those that are not like Powers, have a Similar Composition only by Factors that are not all equal: This is obvious.

(3°.) If the Extremes are not in their least Terms, and yet are like Powers of the Order n, it's manifest from the two preceding Articles, that they are similarly Composite by Factors all equal; and also by such as are not equal.

But observe, for the two last Cases, that it is not easy in all Examples to determine whether the Extremes admit of a similar Composition, by Factors that are all different, or by Factors that are partly different and partly equal; or whether it may not be both Ways. These Things only I find evident. 1. That for a Series of three Terms, whose Extremes are not Squares, the similar Factors must be all different; because there are but two of them; the Invention of which Factors is easy; thus, a, b, being the least Terms of the Ratio, a^2, b^2, are the Extremes in their least Terms; and therefore in the present Supposition, the Extremes are Equimultiples of these, *viz.* $a^2 \times m = a \times am$, and $b^2 \times m = b \times bm$. Again, 2. For a Series of four Terms, where the Extremes are not least in their Ratio, and yet are Cubes, these have similar Factors, either all three different, or two of them equal, and the other different from these: The Reason of which, and the Invention of these Factors, you will easily understand, thus; it has been shewn, that such a Case happens only when the Series is raised, after the Manner of *Problem* I. *B.* IV. *Ch.* IV. from such Terms of the Ratio which are not the least. Suppose then the least Terms of the Ratio, $a : b$, and the Terms from which the Series is raised, to be am, bm (for they must be Equimultiples of the former) then are the Extremes, $\overline{am}^3$, $\overline{bm}^3$; *that is*, $a^3 \times m^3$, $b^3 \times m^3$, which are resolvable into these Factors, *viz.* $a^3 \times m^3 = a \times am \times am^2$, and $b^3 \times m^3 = b \times bm \times bm^2$, similiar to the former; or thus, $a^3 \times m^3 = a \times a \times am^3$, and $b^3 \times {}^3m = b \times b \times bm^3$, similar with the former.

(4°.) In all Cases where the Extremes are like Powers, yet not in their least Terms, and the Number of Terms more than four, they are resolvable into similar Factors that are not all equal, these two Ways, *viz.* 1. Having all the Factors equal, except one, as shewn in the first Demonstration of the *Theorem*: Or, 2. By Factors which are not all equal except one, yet not all different. This last will easily appear, *thus*; The Extremes, according to the Circumstances supposed, are to be expressed $a^n m^n$ and $b^n m^n$, which are resolvable in this Manner, *viz.* $a^n m^n = a \times am^2 \times a \times am^2 \times$ &c. and $b^n m^n = b \times bm^2 \times b \times bm^2 \times$ &c. taking so many Factors in this Manner till a and m, in the one, and b, m, in the other, are as oft involved, as the Index n expresses; or they may be variously resolved, as *thus*, $a^n m^n = a \times a \times am \times am^3$ &c. and $b^n m^n = b \times b \times bm \times bm^3$ &c.

Schol. 2*d*. This *Theorem* is a kind of Reverse to *Theor.* IX. *B.* IV. *Ch.* IV.

Theorem XXXV.

If betwixt any two Integers there falls a certain Number of Means, they must necessarily be all Integers.

$$A : \frac{AM}{N} : \frac{AM^2}{N^2} : \frac{AM^3}{N^3}, \&c.$$

Demon. Let the first Term of a Series be A, and the Ratio in its least Terms be N : M; then by the common Rules, the Series will be expressed as in the Margin. Now the Extremes being Integers, by Supposition; suppose the last of them to be $\frac{AM^n}{N^n}$; then does N^n measure AM^n (else that Extreme is not Integer) but it cannot measure M^n, since 'tis Incommensurable to it; because N is so to M (*Theor.* XIII. *Coroll.*) wherefore it must measure A (*Theor.* VI.) consequently all the inferior Powers of N measure A: And again,

 they

they also measure all the Multiples of A, *i. e.* the Denominators of all the middle Terms, being Powers of N, inferior to N^n, do measure all their Numerators, which are Multiples of A. Therefore, lastly, all these middle Terms are Integers.

COROLL. If in the Progress of a Series, beginning with an Integer, there comes a Fractional Term, simple or mix'd, there can never after that be any more Integers in it; because, if that could happen, then betwixt two Integral Extremes a Fractional middle Term might happen, contrary to what has been demonstrated.

§. III. *Containing a Variety of* Problems, *concerning* Geometrical Progressions, *considered with Regard to their Terms being* Integral *or* Fractional; *whose Solutions depend upon the preceding Doctrine.*

IN a *Geometrical Series* there may be a Variety of Changes from Integers to Fractions [proper or improper] or from these to those; all which depend upon the Relation of the first Term of the Series to the Ratio; and of these, considered also by themselves; as whether the first Term is an Integer or Fraction, and whether the Ratio is Multiple or not. From whence arises a new Set of *Problems*, relating to these Series; which have been referred to this Place, because the Demonstrations depend upon the Composition of Numbers by their Primes. I shall begin with explaining all the various Changes that can be in a Series.

PROBLEM VII.

It is required to shew all the Variety of Changes from Integers to Fractions, and mix'd Numbers, and from these to those, that can possibly be in any Series of Geometrical Proportionals; and to give Rules for the Invention of Series under all the possible Variety.

SOLUTION.

This Complex Problem may be resolved into two Parts, as we consider the Series to increase or decrease: Yet we shall have only one of the Parts to demonstrate; because the Variety of the one is comprehended in the other. I shall first explain the Variety in an increasing Series, and the other will easily be seen in that: But in order to this, there is one general Proposition relating to both Kinds; which being of Use in the particular Parts of the Problem, I shall premise as a

LEMMA.

If an increasing Series begins with a Proper Fraction ever so small; if it's a finite or determinate one, and the Ratio also determinate; then, after a certain Number of Terms continued in Fractions, it will increase to a whole or mix'd Number, and that too, greater than any assignable Number.

Again, Let a decreasing Series begin with a Number ever so great; if it's finite, and the Ratio so also; it will decrease to a proper Fraction, and that too, less than any assignable one. The Reason of all which is evident and needs no Demonstration; but if any call for it, they will find it afterwards in *Theor.* I. and II. *Ch.* III.

We proceed to the *Solution* of the *Problem.*

PART I. *For an Increasing Series.*

Case I. *If the first Term of a Series is an Integer, the Varieties are these;*

1°. It may continue in Integers; which necessarily follows from a Multiple Ratio: For it's plain, that the Product of two Integers will be an Integer. That the Series can-

not

not continue in Integers, if the Ratio is not Multiple, the next Article will ſhew,

2°. It may change into a mix'd Number; and there it will continue; which requires and follows from a Ratio not Multiple; as in this *Example*, $4 : 6 : 9 : 13\frac{1}{2} : 20\frac{1}{4}$, &c. the Ratio being $2 : 3$.

$$A : \frac{AM}{N} : \frac{AM^2}{N^2} : \frac{AM^3}{N^3} : \&c.$$

Demon. The leſſer Extreme of a Series being A, and the Ratio, in its leaſt Terms, N : M, the Series is expreſſed as in the Margin.

Now, as the Denominator ſtill increaſes in every Term in a determined Ratio, *viz.* $1 : N$; ſo it will become neceſſarily, after a certain Number of Terms, greater than A (by the preceding *Lemma*) and the firſt Term wherein that happens (and conſequently all above) will be a mix'd Number; let that Term be $\frac{AM^n}{N^n}$. I ſay, N^n does not meaſure AM^n; for N, M are Incommenſurable; and ſo are N^n, M^n (by *Coroll. Theor.* XIII.) But if N^n meaſures AM^n, it will alſo meaſure A; (*Theor.* VI.) *i. e.* a greater will meaſure a leſſer, which is abſurd: Therefore N^n does not meaſure AM^n; or $\frac{AM^n}{N^n}$ is a mix'd Number: For the ſame Reaſon all the Terms above are mix'd.

But again, *Obſerve*, that there may be a mix'd Number before the Denominator is greater than A; as in this *Exam.* $6 : \frac{6\times 3}{2} = 9 : \frac{6\times 9}{4} = 13\frac{1}{2}$: And leſt it be ſuppoſed, that after a mix'd Number, which comes into the Series before N^n is greater than A, there may come an Integer; we ought to demonſtrate univerſally, that after a mix'd Number, there can never be an Integer; and this you have already ſeen, in *Cor. Theor.* XXXV. Where you alſo ſee, that from an Integer there can be no more Changes but into a mix'd Number.

Or, the whole Demonſtration may be made more ſimply; *thus*, It's evident we can chuſe a Number, N, which is Incommenſurable to A; and being ſo alſo to M, no Power of N can meaſure the Product of A into any Power of M; becauſe theſe are Incommenſurable (*Theor.* X.) Hence none of the Terms of the Series, after A, will be Integers.

Caſe II. *The firſt Term being a mix'd Number, the Varieties are,*

$$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \frac{AM^3}{B}, \&c.$$

1°. It may continue in mix'd Numbers for ever; which is neceſſarily effected, either, 1. By a Multiple Ratio, $1 : M$, provided M be ſuch, as to be Incommenſurable with B: Or if B has any Prime different from any of the Primes of M, or any the ſame Primes, oftner involved; for then, by what's ſaid in the laſt Caſe, B can never meaſure any of the Numerators. 2. By a Ratio not Multiple, as N : M, provided N has in its Compoſition any Prime which is not in A; or any the ſame Prime oftner involved; or if B has any Prime which is not in M: For in any of theſe Circumſtances, none of the Denominators can ever meaſure its Numerator.

$$\frac{A}{B} : \frac{AM}{BN} : \frac{AM^2}{BN^2} : \frac{AM^3}{BN^3} : \&c.$$

Exam. 1. $3\frac{1}{2} : 10\frac{1}{2} : 31\frac{1}{2} : 94\frac{1}{2}$; the Ratio being $1 : 3$.

Exam. 2. $\frac{8}{3} : \frac{40}{12} : \frac{200}{48} : \frac{1000}{192}$; the Ratio being $4 : 5$.

 2°. It

$$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \&c.$$

2°. It may pass from a mix'd Number into an Integer, and so continue for ever; which can be effected only by a Multiple Ratio, 1 : M, such, that M have in its Composition all the Primes that compose B. I shall first prove that such a Ratio will bring the Series to an Integer; which is thus evident: M having all the Primes that are in B; if they are also as oft involved, then it's plain, that B measures M; and consequently $\frac{AM}{B}$ is an Integer. Again, However oftner the same Primes are involved in B than in M, yet being more and more involved in the higher Powers of M, there must be a Power in which they are all oftner involved than in B; and hence it's plain, that there will at last be a Term in the Series in which B measures the Power of M in the Numerator, therefore that Term must be Integral.

Again, after an Integer comes into the Series, it must continue for ever in Integers, because the Ratio is Multiple.

In the next Place, I shall demonstrate, that no other Conditions will produce this Variety: And, (1.) A Ratio not Multiple, though it could bring the Series to an Integer, yet it could not continue so, as has been proved in *Case* I. *Art.* 2. (2.) If M has not in it all the Primes of B, it cannot measure B (*Theor.* X.) and hence also B will never measure any Power of M, because M^n has no other Primes than M, (*Cor.* 5. *Theor.* X.) And since B is Incommensurable to A, therefore (by *Cor.* 2. *Theor.* VI.) it cannot measure AM^n, that is, none of the Terms can ever be Integers.

Exam. $5\frac{1}{8} : 14\frac{7}{12} : 112\frac{1}{2} : 675 :$ &c. the Ratio 1 : 7

Observe, Examples of this Variety may be invented more simply, thus, *viz.* Take any Integer, and any Multiple Ratio, whose greater Term is less than the assumed Integer, and yet is not an *aliquot* Part of it; by this you'll find at least one mixt Term below that Integer; from which again the Series will proceed upwards in the manner proposed. But the other Method shews the fundamental Reason of the Case from the Composition of Numbers; and is of Use in some of the following Problems.

(3°.) It may pass from a mixt Number into an Integer, and from that again into a mix'd Number; and so continue; which can be effected only by a Ratio not Multiple, as N : M; such, that M do contain in its Composition all the different Primes of B, and A all those of N, which must also be as oft at least involved in A as in N. The Reason of which is this;

$$\frac{A}{B} : \frac{AM}{BN} : \frac{AM^2}{BN^2} : \frac{AM^3}{BN^3} : \&c.$$

if the Ratio were Multiple, then either the Series would never pass into an Integer, or if it did, it would continue so (by *Case* II.) Again, if there be any Prime in B, which is not in M; or in N, which is not in A, it is manifest (from *Theor.* X.) that none of the Divisors, as BN^n will ever measure the Dividend AM^n. Also, the Primes of N must not only be all involved in A, but they must be at least as oft involved in A as in N; for, if any of them is oftner in N, it will be much more so in the higher Powers of N; and hence (by *Theor.* X.) none of the Denominators can ever measure its Numerator, *i. e.* none of the Terms will ever be an Integer. These are the general Conditions: But yet more particularly, if any of the Primes of M are oftner involved in B; then conceive, that one of them whose Index in B exceeds its Index in N, by the greatest Difference; also conceive the least Number, which multiplying the Index of that Prime in M will produce a Number not less than its Index in B; then must N be such, that taking the Prime, whose Index in N and A differ least, and multiplying its Index in N by the same Multiplier, the Product shall not be greater than its Index in A. The Reason of which is also plain, for otherwise, before we come to a Term in which

which the Power of M contains all the Primes of B as oft involved, which is neceſſary to make an Integer, the like Power of N in the Denominator will contain all the Primes of A as oft involved; and therefore in all the Terms above they will be oftner involved in N, and conſequently the Denominator will never meaſure the Numerator, *i.e.* there will never be an Integer in the Series.

Theſe Conditions are plain enough to ſhew the Invention of ſuch a Series; whereof take this Example, $\frac{56}{45} : \frac{840}{90} : 70 : 525 : \frac{7875}{2}$: the Ratio being $2 : 15$; and that you may compare it with the preceding Conditions, the Compoſition of the firſt Term, repreſented by $\frac{A}{B}$, is this, $\frac{56}{45} = \frac{2\times2\times2\times7}{3\times3\times5}$, and the Ratio, $\frac{M}{N}$ is $\frac{15}{2} = \frac{3\times5}{2}$.

Theſe are all the Changes from a mixt Number, becauſe when a mixt Number follows Integers, the Series can never paſs again into Integers; as has been ſhewn in *Art.* 2. *Caſe* I.

Caſe III. *The firſt Term being a proper Fraction, the Varieties are theſe.*

[1°.] It may paſs into a mixt Number; and continue ſo; which is effected by a Ratio Multiple or not Multiple, qualified as in *Variety* I. *Caſe* II; for $\frac{A}{B}$ may be either a proper or improper Fraction; and ſince with the Qualifications refer'd to there can never be an Integer in the Series, and by the premiſed Lemma it muſt encreaſe either to a whole or mixt Number; therefore it will paſs into a mixt Number, and there continue.

Exam. 1. $\frac{2}{13} : \frac{6}{13} : \frac{18}{13}$, &c. the Ratio $1 : 3$.

Exam. 2. $\frac{3}{29} : \frac{15}{58} : \frac{75}{116} : \frac{375}{232}$, &c. Ratio $2 : 5$.

[2°.] It may paſs into a whole Number, and there continue; which requires a Multiple Ratio, qualified as in *Var.* 2. *Caſe* II. with this further, that ſuppoſing M^n the leaſt Power of M which B meaſures, the Products of all the inferior Powers of M multiplied into A, muſt be leſs than B; elſe it's plain, that it will paſs firſt into a mixt Number.

Examples are eaſily invented by taking M equal to B, or to any Multiple of it; or taking any Integer A, and a Multiple Ratio $1 : M$, ſuch, that M is greater than A; for then continuing the Series downwards, it will fall firſt into a proper Fraction, from which it proceeds upwards in the Manner propoſed.

Exam. $\frac{2}{9} : \frac{2}{3} : 2 : 6 : 18$, &c.

[3°.] It may paſs firſt into a mixt Number, and then into an Integer, and there continue; which requires a Multiple Ratio qualified as in *Var.* 2. *Caſe* II. with this further, that ſuppoſing M^n to be the leaſt Power of M which B meaſures, ſome of the next inferior Powers multiplied by A ſhall produce a Number greater than B; which is eaſily invented; or may be had by taking any *Example* of *Var.* 2. *Caſe* II. and continuing it downwards; for it muſt firſt paſs into a proper Fraction, ſince there cannot be an Integer both above and below a mixt Number in the ſame Series.

Exam. $675 : 112\frac{1}{2} : 14\frac{7}{12} : 3\frac{1}{8} : \frac{25}{56}$, &c. Ratio $7 : 1$.

[4°.] It may paſs firſt into a whole Number, and then into a mixt, and ſo it muſt continue, by Virtue of *Theor.* XXXV; and this is effected by a Ratio not Multiple,

ple, qualified as in *V.* 3. *Case* II. with this further, that supposing $\frac{M^n}{N^n}$ the least Power of $\frac{M}{N}$, which multiplied by $\frac{A}{B}$ makes an Integer, all the inferior Powers produce a proper Fraction. The Invention of Examples is easy, by assuming any Integer A, and a Ratio not Multiple, N : M, such, that NA be less than M; for then $\frac{AN}{M}$ is a proper Fraction, and $\frac{AN}{M} : A : \frac{AM}{N} : \frac{AM^2}{N^2}$, &c. represents the Series required.

Exam. $\frac{12}{17} : 4 : \frac{68}{3}$: &c. Ratio 3 : 17.

[5°.] It may pass first into a mixt Number, then into an Integer, and lastly, into a mixt Number; which requires a Ratio not Multiple, qualified as in *V.* 3. *Case* II. with this further, that supposing $\frac{M^n}{N^n}$ the least Power of $\frac{M}{N}$, which multiplied into $\frac{A}{B}$ produces an Integer, some of the next inferior Powers shall produce a mixt Number; Examples of which are invented by taking an Example of *V.* 3. *Case* II. which being continued downwards must fall into a proper Fraction, but never into a whole Number, because there is a whole Number above the mixt.

Exam. $\frac{112}{675} : \frac{56}{45} : \frac{840}{90} : 70 : 525 : \frac{7875}{2}$, &c. Ratio 2 : 15.

PART II. *For a decreasing Series.*

The Varieties here are included in the former, and therefore I need only to name them, thus,

Case I. *The first Term being an Integer, then*

1°. It may pass first into a proper Fraction, and so continue.
2°. It may pass first into a mixt Number, and then into a proper Fraction.

Case II. *The first Term being a mixt Number.*

1°. It may pass first into a proper Fraction and continue so.
2°. It may pass first into a whole Number, and then into a proper Fraction.
3°. It may pass first into a whole Number, then into a mixt, and lastly, into a proper Fraction.

Case III. *The first Term a proper Fraction.*

It can only continue in Fractions.

SCHOL. In the following Problems, a given Ratio, or the same Ratio, do always comprehend a Ratio with its reciprocal; either of which is to be supposed, as we take a Series to encrease or decrease, and we suppose a Ratio always in its least Terms.

PROBLEM VIII.

Any whole Number being given, with any Ratio, to find how many Integral Terms can possibly be joined in the same continued Series with A, taking it either encreasing or decreasing, or both Ways, in that given Ratio.

SOLUTION.

We must consider this Problem in two Parts, according as the given Ratio is Multiple, or not.

Case

Case I. The given Ratio Multiple. Then it's plain that the Series proceeds encreasing *ad Infinitum* in Integers; but is limited decreasing. Thus; let the given whole Number be A, and the Ratio M : 1; then if M is greater than A, there can be no Integral Term added; if M=A, then there will only be one which will be 1; but if M is less than A, then the Number of Integral Terms that can be added is equal to the Index of the highest Power of M which measures A; and if M does not measure A, there can no Integer be added; all which is manifest in this general Expression of the Series, $A : \frac{A}{M} : \frac{A}{M^2} : \frac{A}{M^3} :$ &c. But this Solution requiring, that every Term of the Series be actually raised, we may solve it otherwise; *thus*, resolve A and M into all their Primes; and if there is any Prime in M, which is not in A, or any the same Prime oftner involved, then M does not measure A, and so there will never be an Integral Term after A: But if A contains all the Primes of M, and all of them at least as oft involved; take that Prime whose Index in A exceeds its Index in M by the least Difference; also seek the greatest Number, which multiplying the Index of that Prime in M, makes a Number not greater than its Index in A, that Number is the Number sought. The Reason of which is obvious, for these Terms will always be whole Numbers, till some one at least of the Primes of A is oftner involved in the Power of M, which is the Denominator; and it's evident that this will happen first to that Prime, the Index of whose Involution in M comes short of its Index in A by the least Difference; and since any Power of M, as M^n, is the Product of the like Powers of all the Primes of M, the Truth of the Rule is manifest.

Exam. Let 72 be the given Number, and the Ratio 4 : 1; then taking the Primes of 72 and 4, it is $72=2\times2\times2\times3\times3$, and $4=2\times2$. Now there being but one Prime in 4, whose Index is 2, and the Index of the same Prime in 73 being 3, I find that 1 is the greatest Number, which multiplying 2, the Index of 2 in 4, produces a Number not greater than 3, the Index of 2 in 72; therefore there can be but 1 Integral Term added, as here, $72 : 18 : 4\frac{1}{2} :$ &c.

Case II. The Ratio not Multiple. It's plain that the Number of Integral Terms is limited, both encreasing (by *Part* I. *Case* I. of the preceding *Probl.*) and decreasing (by the preceding *Lemma*;) and to find the Number, take it first encreasing, and suppose the given Number A, and the Ratio in its least Terms to be N : M; the Index of the greatest Power of N, which measures A, is the Number sought.

$$\frac{AN^3}{M^3} : \frac{AN^2}{M^2} : \frac{AN}{M} : A : \frac{AM}{N} : \frac{AM^2}{N^2} : \frac{AM^3}{N^3} : \text{\&c.}$$

The Reason of which is this; suppose the last Integral Term to be $\frac{AM^n}{N^n}$, so that N^n measures AM^n, but it does not measure M^n, they being Prime to one another, because N and M are so; therefore, by (*Theor.* VI.) N^n does measure A; but it is also the greatest Power of N that does so; for if a greater does, as N^{n+1}, then that will also measure AM^{n+1}; and so $\frac{AM^n}{N^n}$ is not the last Integral Term, contrary to Supposition. *Lastly*, Because the Index of the Powers of N shew their Distance after A, therefore the Rule is true.

2°. If the Series is to be taken decreasing, then the Index of the greatest Power of M, which measures A, is the Number sought; for the same Reason as was explained in the preceding Article.

Observe, That this Case may be solved the same Way as *Case* I. by comparing the Primes of A with those of N for the encreasing Series, and with those of M for the decreasing.

3°.

3°. If it's proposed to find the greatest Number of Integral Terms that can be joined in the same Series with A in the Ratio N : M, taken both encreasing and decreasing; then find separately how many can be added encreasing in the Ratio N : M, and how many decreasing in the Ratio M : N; the Sum is plainly the Number sought.

Exam. Let the given Number be $72=2\times2\times2\times3\times3$, and the Ratio be $2:3$; then is the greatest Number of Integral Terms that can be joined with it, 5, *viz.* 3 above and 2 below, making this Series, $21\frac{1}{3} : 32 : 48 : 72 : 108 : 162 : 243 : 364\frac{1}{2}$. For 8 the third Power of 2 is the greatest Power of it that measures 72, and 9 the second Power of 3 is its greatest Power that measures 72.

Schol. We may make the preceding Problem yet more general and unlimited, by supposing no particular given Ratio; but proposing to find the greatest Number of Integral Terms that can possibly be joined in the same Series with a given Integer in any Ratio whatever; the Answer to which is an infinite Number, because with a Multiple Ratio the Series goes on encreasing *ad infinitum*; but if we take these Limitations, *viz.* 1°. A Ratio Multiple and a decreasing Series; then though no particular Ratio is given, the Problem may be solved. *Thus*; resolve the given Number, A, into its Primes, the Index of that one which is oftnest involved is the Number sought, and that Prime it self is a Ratio which will make the Series. *Example*, If $72=2\times2\times2\times3\times3$ is the given Number, then you can join in the same Series decreasing, only three Integral Terms, 3 being the Index of 2 in 72, and the only Ratio that can effect this is $2:1$, which makes this Series $72 : 36 : 18 : 9 : 4\frac{1}{2}$. *Observe also*, That if there are more than one of the Primes of A involved to the same Power, which is the highest of any concerned in it, then any of these Primes, or the Product of any two or more of them, may be made the Ratio; so if $216=2\times2\times2\times3\times3\times3$ is the given Number, the Answer is also 3; and the Series may be made in any of these Ratios, and in these only, *viz.* $2:1$. or $3:1$: or $6:1$, making these different Series $216 : 108 : 54 : 27$: or $216 : 72 : 24 : 8$. Or, *Lastly*, $216 : 36 : 6 : 1$. Thus we learn, not only how to solve the Problem, but to find also all the Variety of Ratios that can possibly solve it. The Reason of which is obvious.

Again, 2°. We shall suppose the Problem limited to a Ratio not Multiple, without any determined or given one; then is the Solution made thus, resolve A into its Primes, and take these two, the Sum of the Indexes of whose Involution in A, is the greatest, that Sum is the Number sought; and these Primes are the Terms of a Ratio, in which the Series may be made; and if several of these Sums which are the greatest, are equal among themselves, then the Primes to which they correspond, or the Products of any two or more of them, make all the different Ratios in which the Series can be made. *Exam.* Let the given Number be $3024=2\times2\times2\times2\times3\times3\times3\times7$, then is the Number sought 7, the Sum of the Indexes of 2 and 3, *viz.* 4 and 3, which is greater than the Sum of any other two Indexes; and therefore the Ratio is $2:3$, which in this Example is the only one that can make a Series of 7 Integers joined to 3024; which Series is this, $896 : 1344 : 2016 : 3024 : 4536 : 6804 : 10206 : 15309$.

3°. If it's proposed only to find the greatest Number of Terms that can be joined, increasing or decreasing, separately considered; then take the Index of the Prime, which is oftnest involved in A, and that is the Number sought for an encreasing Series; that Prime, or any other as oft involved, or the Product of any two or more of these, being the Antecedent or lesser Term of the Ratio, which can produce the Series, to which we may take any other greater Number, which is Prime to it for a Consequent. Then for a decreasing Series we must chuse that Prime whose Index is the greatest, that has another Number lesser than it self, and Prime to it. So in the preceding Example, 3024, the greatest Number encreasing is 4, the Index of 2 in 3024; and in this Case there is an infinite Choice of Ratios, because we may take any

any Number greater, and Prime to 2, for a Consequent; but for a decreasing Series, the Number is only 3, the Index of 3 in 3024, and there is but one Ratio, *viz.* 3 : 2, which will produce the Series, because there is but one Number, *viz.* 2, which is lesser than 3, and Prime to it. Take this other *Example*, $9261 = 7 \times 7 \times 7 \times 3 \times 3 \times 3$. To this can be joined at most 3 Terms encreasing, the Index of 7 or 3 being 3; and the Series may be produced by any Ratio, whose Antecedent is 7 or 3, or 21. Again, decreasing there can at most be added 3, which also can be effected by any Ratio whose Antecedent is 3 or 7, or 21; which makes a limited Number of different Ratios, because there is a limited Number of Terms lesser than these Antecedents and Prime to them, which are these, 3 : 2, 7 : 5, 7 : 3, 7 : 2; and if 21 is made the Antecedent, the Consequents are these, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 19, 20. But the absolutely greatest Number of integral Terms that can stand in the same Series with 9261 is 6, whereof 3 will be greater, and 3 lesser, in the Ratio 3 : 7, which is the only one that will make the Series.

PROBLEM IX.

A whole Number being given, to find a Ratio in which it is possible to join a given Number of integral Terms, and no more, in the same Series with the given whole Number.

SOLUT. If we take the given Number for the greater Extreme of a Series, then the Problem being possible in the Circumstances of the given Numbers, may be solved in some Cases by a Multiple Ratio only; in others by a Ratio not Multiple and in others by both: But if the lesser Extreme is given, it requires a Ratio not Multiple; which it does also, if we consider the given Number as one of the middle Terms. *Thus*,

$$A : \frac{A}{M} : \frac{A}{M^2} : \&c.$$

1°. Let us take the given Number, A, for the greater Extreme of a Series, then resolve it into its Primes; and if the Index of any of its Primes is precisely equal to the given Number of Terms, that Prime, or the Product of any two or more such, or any such Product multiplied again into any of the Primes which have a greater Index, is the greater Term of a Multiple Ratio which solves the Problem. And if there is any Number less than it, and Prime to it, that being made Consequent, makes a Ratio not Multiple, which also solves the Problem. But if none of the Primes of A has an Index exactly equal to the given Number of Terms, the Problem is impossible, making the given Number the greater Extreme of the Series.

Exam. Given 567, and the Number of Terms 4, then the Ratio sought is 3 : 1; for $567 = 3 \times 3 \times 3 \times 3 \times 7$, and there is not another Ratio that will answer; but take $189 = 3 \times 3 \times 3 \times 7$, and the Problem is impossible.

2°. The given Number being the lesser Extreme, find the Antecedent of the Ratio sought the same way, as in the preceding Case; and for a Consequent, any Number greater, and Prime to it: So 567 being the lesser Extreme, and the Number of Terms 4, the Ratio sought must take 3 for the Antecedent, and any Number greater, and Prime to it, for the Consequent.

3°. If the given Number is considered as one of the middle Terms indefinitely, then take these two Primes of A, the Sum of whose Indexes is equal to the given Number of Terms, and these are the Terms of a Ratio, which solves the Problem; and of the Terms to be added, there will be as many greater than the given Number, as the Index of the lesser one of these two Primes, and as many decreasing as the Index of the greater.

Exam. Given $72 = 2 \times 2 \times 2 \times 3 \times 3$, and the Number of Terms 5, the Ratio is 2 : 3, the Number of Terms greater being 3, and the lesser being 2; and in this Case there is no other Variety in the Solution. But suppose $3528 = 2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7$, and the

the Number of Terms 5, it may be solved by this Ratio 2 : 3 or 2 : 7, with 3 Terms greater and 2 lesser; and if the Number of Terms is 4, then it is solved by this Ratio 3 : 7 with 2 Terms greater, and 2 lesser, or with this Ratio 6 ($=2\times3$) : 7 with 2 greater and 2 lesser. *Observe*, If the Sum of no two of the Indexes is exactly equal to the given Number of Terms, then the given Number cannot stand as a middle Term in the proposed Circumstances.

The Reason of all this is plain, especially if it be compared with the preceding Problem.

Schol. If the proposed Number of integral Terms is *o*, That is, to find a Ratio with which it's impossible to join any integral Term to a given Integer, we have no more to do, but, take any two Numbers which contain a Prime different from any in, A, or any the same Prime oftner involved, and make these the Terms of your Ratio; for then it's plain, that in such a Ratio you can join no integral Term, either greater or lesser, because none of these Numbers can measure A.

Again, if the Ratio is required to be such, that the Number of integral Terms to be added is infinite, then take any Multiple Ratio.

PROBLEM X.

To find a whole Number, with which a given Number, and no more, of integral Terms can be joined in the same Series in a given Ratio.

Solut. 1°. If the Ratio is Multiple, the Problem is plainly impossible; for to any Integer an infinite Number of integral Terms may be added in any Multiple Ratio, yet if we limit it so as to consider the Number sought as the greater Extreme of a Series, it is solved thus; if the greater Term of the given Ratio has any of its Primes involved to an Index precisely equal to the given Number of Terms, then any Number which is a Multiple of that Power of that Prime, is the Number sought; but if none of its Primes is so oft involved, then is the Problem also impossible.

2°. If the Ratio, is not Multiple, raise its Terms to the Power whose Index is the given Number of Terms less 1; and these are the Extremes of a Series in that Ratio, having the given Number of Terms, and to which it is impossible to add another in Integers (*Theor.* XXIII.) And these Powers, or any of the middle Terms of the Series, whereof they are the Extremes, which are easily found (vid. *Probl.* VI. *Schol.* I.) are Numbers which solve the Problem, and are indeed the only Numbers that do so. And *observe*, if it's proposed to find the Number sought, so that it shall be any one of the Extremes, or any one of the middle Terms, it's easily done by what's shewn.

Exam. The Ratio 2 : 3, and Number of Terms 6; I take the 5th Power of 2 : 3, *viz.* $2\times2\times2\times2\times2=32$, and $3\times3\times3\times3\times3=243$, and these are Extremes of the Series; the middle Terms being composed of the Powers of these, according to this Form A^5 : A^4B : A^3B^2 : A^2B^3 : AB^4 : B^5.

Scholiums.

I. If the proposed Number of Terms is *o*, then with a Multiple Ratio the Problem is impossible, if the given Number is to be made the lesser Extreme; but if we make it the greater, then we may take any Number lesser than the Antecedent of the Ratio, or greater, provided it has not all the different Primes of the given Antecedent, or has some of them less involved.

Again, with a Ratio not Multiple, we have no more to do but take any Number which has not in its Composition all the Primes of either of the Terms of the Ratio, or has any one of each of them less involved.

II.

II. If there is no particular Ratio given, nor any Limitation, *i. e.* if we demand an Integer, with which can be join'd only a certain Number of integral Terms in any Ratio, then the Problem is impossible, because in a Multiple Ratio there may be an infinite Number of Terms encreasing; but suppose these general Limitations,

1°. That the Number sought be the greater Extreme of a Series; then it may be found, thus; of any prime Number take that Power whose Index is the given Number of Terms; or take any Multiple of such a Power, provided it has no Prime involved to a higher Power; and to that Number can be joined the given Number of integral Terms decreasing, in any Ratio whose Antecedent that Prime is.

Exam. Let the Number of Terms be 3, then 8 is the Number sought, and the Ratio 2 : 1. Or $8\times5=40$, and the Ratio 2 : 1; but 16 is not such a Number, because it admits of 4 Terms in the Ratio 2 : 1. Again, $27=3\times3\times3$ is a Number answering the Problem, and the Ratio is either 3 : 1, or 3 : 2, making these Series 27 : 9 : 3 : 1, or 27 : 18 : 12 : 8.

2°. Suppose the Limitation is to any Ratio not Multiple; then take, as before, that Power of some Prime, whose Index is the given Number of Terms, and that is the Number sought; the said Prime being the Antecedent of a Ratio which solves the Problem, encreasing, any greater Number and Prime to it being made the Consequent. Also, if there is any Number lesser than it, but greater than 1, that being made Consequent will be a Ratio which will solve it decreasing. *Lastly,* If you take any two Primes, and involve them each to such a Power, that the Sum of their Indexes be equal to the given Number of Terms, then the Product of these Powers is a Number to which the given Number of Terms, and no more, can be added, taking part of them greater, and part lesser; which may also be distributed in any Proportion; for whatever Index the lesser Prime is raised to, that will express the Number of Terms greater than the Number sought, and the Index of the other is the Number lesser, these two Primes being the Terms of a Ratio that answers the Problem; and in this Case the given Number of Terms is the greatest Number of integral Terms absolutely that can be joined with the Number sought.

Exam. Given Number of Terms 5. I take $3\times3\times3\times3\times3=243$, which is a Number to which can be joined 5 integral Terms, and no more increasing, the Ratio of which must have 3 for its Antecedent, and any Number greater, and Prime to it, for a Consequent as 4; and decreasing it can also admit 5 Terms, but limited to this one Ratio not Multiple, *viz.* 3 : 2, because 2 is the only lesser Number prime to 3. Again, 108 $(=3\times3\times3\times2\times2)$ can have joined with it only 5 integral Terms in the Ratio 2 : 3, whereof 2 will be greater and 3 lesser, as here, 32 : 48 : 72 : 108 : 162 : 243.

III. If it be proposed to find an Integer, to which it is impossible to join any Integer in the same Series, in any Ratio, the Problem is impossible; because to any Integer can be joined at least another Integer.

PROBLEM XI.

Any Fraction, proper or improper, $\frac{A}{B}$, being given, to find how many fractional Terms can possibly be joined with it in the same Series.

SOLUTION.

If there is no further Limitation of this Problem, the Answer is an infinite Number; for if the Fraction is proper it admits an infinite Number of fractional Terms, decreasing in the same Series in any Ratio whatever; and if it's an improper Fraction, then also an infinite Number of fractional Terms can be joined encreasing or decreasing, or both Ways; for the Series being continued both Ways, there can't be one Integer

above and another below; (by *Theor.* XXXV.) therefore the Series will continue in Fractions one or both Ways.

But as the Number sought may be limited on one Side (though it can't on both) we shall enquire upon what Side it is limited, and to what Number of Terms. Thus,

Case I. *The given Ratio Multiple, as* 1 : M, *Then,*

(1°.) The Series will continue for ever in Fractions decreasing: If the given Fraction is proper, this is manifest; and if it's improper, you see the same Truth from this Consideration, *viz.* That if ever there comes an Integer into the Series, from that reversely it will continue upwards to a mixt Number (from whence it proceeded downwards) which is impossible in a Multiple Ratio, because every Term is the Product of two Integers. Therefore there can no Integer come into the Series. But,

(2°.) Take the Series encreasing from the given Fraction, and the Number of fractional Terms that can be added, will in some Cases be finite, and in some infinite, which the following Rule will discover, whether the given Fraction is proper or improper; *Thus*, resolve M and B both into their Primes; and if there is any Prime in B which is not in M, then the Number of Terms required is infinite; for B can never measure any Power of M, and being Prime to A, it cannot measure any of the Numerators (*Cor.* 2. *Theor.* VI.)

$$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \&c.$$

Again, if all the different Primes of B are found in M, and as oft at least involved; then is B an *aliquot* Part of M, and hence $\frac{AM}{B}$ is an Integer, therefore there can no fractional Term be added. *Lastly*, If all the Primes of B are in M, and some of them also oftner involved in B than in M, then take all of these whose Indexes in B and M have the same Difference, and that, the greatest of any other of them; and chusing that one of these, whose Index is greatest, find the greatest Number, which multiplying its Index in M, will make a Product less than its Index in B; that Number is the Number sought; for it's the Index of the greatest Power of M which B does not measure, since B measures none of the Powers of M till you come to one which has all the Primes of B, at least as oft involved; but any Power of M is the Product of the like Powers of all its Factors (as follows from *Theor.* I. *Book* III. *Chap.* I.) Hence the Truth of the Rule is clear.

Exam. 1. $\frac{2}{15}$ or $\frac{37}{15}$, and the Ratio 1 : 7; there can never be any integral Term, because all the Primes of 15, which are 3, 5, are different from 7.

Exam. 2. $\frac{2}{3}$ or $\frac{7}{3}$, and the Ratio 1 : 6. There can be no fractional Term added, because 3 is an *aliquot* Part of 6, whereby the very next Term will be an Integer.

Exam. 3. $\frac{7}{24}$ or $\frac{75}{24}$, and the Ratio 1 : 6. Here $24 = 2 \times 2 \times 2 \times 3$ and $6 = 2 \times 3$; then the Index of 2 in 24 is 3, and in 6 it is 1, and the greatest Number, which multiplying 1 will produce a Number less than 3 is 2; therefore there can be but 2 fractional Terms joined with the given one; as here, $\frac{7}{24} : \frac{42}{24} : \frac{252}{24} : \frac{1512}{24} = 63$: And here, $\frac{75}{24} : \frac{450}{24} : \frac{2700}{24} : \frac{16200}{24} = 675$.

Case II. *The given Ratio not Multiple, as* N : M, *which we shall again subdivide according as the given Fraction is proper or improper.*

(1°.) If the given Fraction is proper, it's already shewn, that the Series must continue for ever in Fractions decreasing, but will not in all Cases do so encreasing; and the Number sought may be found thus; resolve A, B, N, M, into their Primes, and if there is any Prime in N which is not in A, or the same Prime oftner involved; or again, any Prime in B which is not in M, then the Series will continue for ever in Fractions; because the Denominators can never measure any of the Numerators.

$$\frac{A}{B} : \frac{AM}{BN} : \frac{AM^2}{BN^2} : \&c.$$

But if none of these Circumstances happen, *i. e.* If there is no Prime in B which is not in M, nor in N which is not in A, nor any the same Prime oftner involved in N than in A, then if all the Primes of B are at least equally involved in M, the very first Term after $\frac{A}{B}$ is an Integer. But if any one or more Primes of B are not so oft involved in M; take all of those whose Indexes in B and M have the same Difference, and that too, the greatest of any other of them, and chuse that one of these whose Index is greatest; then find n, the greatest Number which multiplying its Index in M, produces a Number less than its Index in B, and that is the greatest Number of fractional Terms; supposing there can be an Integer; the Reason is easy: And to know if there can be an Integer do this; take all the Primes whose Indexes in N and A have the same Difference, and that, the least of any other, then chuse that one whose Index is greatest; and if the Product of its Index in N, multiplied by $n+1$, produces a Number not exceeding its Index in A, then the $n+1$ Term after $\frac{A}{B}$ is an Integer, and n is the Number sought; but if it's greater, the Series will continue for ever in Fractions.

Exam. 1. $\frac{A}{B}=\frac{21}{55}=\frac{3\times7}{5\times11}$ and $\frac{M}{N}=\frac{65}{22}=\frac{5\times13}{2\times11}$ or $\frac{M}{N}=\frac{98}{43}=\frac{2\times7\times7}{3\times3\times5}$. Here the Series will continue for ever in Fractions.

Exam. 2. $\frac{A}{B}=\frac{42}{55}=\frac{3\times7\times2}{5\times11}$ and $\frac{M}{N}=\frac{275}{21}=\frac{5\times5\times11}{3\times7}$. Here the very first Term after $\frac{A}{B}$ is an Integer, *viz.* $\frac{42}{55}\times\frac{275}{21}$ or $\frac{3\times7\times2}{5\times11}\times\frac{5\times5\times11}{3\times7}=10$, by taking away common Factors, out of the Numerators and Denominators.

Exam. 3. $\frac{A}{B}=\frac{80}{567}=\frac{2\times2\times2\times2\times5}{3\times3\times3\times3\times7}$ and $\frac{M}{N}=\frac{105}{2}=\frac{3\times5\times7}{2}$. Here there will be 3 fractional Terms after $\frac{A}{B}$, which I find thus; 3 is the Prime in B, whose Index exceeds its Index in M by the greatest Difference, and no other have the same Difference; also 3 is the greatest Number which multiplying 1, its Index in M, makes a Product 3 less than its Index in B, which is 4. Again, as there is but one Prime in N; *viz.* 2, whose Index is only 1, this multiplied by 4 $(=n+1)$ produces 4, a Number not exceeding 4, the Index of 2 in A; therefore 3 is the Number sought. But if we suppose A to be $40=2\times2\times2\times5$, then because the Product 4 exceeds 3, the Index of 2 in A, therefore the Series must continue for ever in Fractions, because the $n+1$ Term after $\frac{A}{B}$ is the first that could possibly be an Integer, upon the Consideration of the Primes of B compared with the same in M. But then this is hindered by the Consideration of the

the Primes of N, compared with the same in A. For at the $n+1$ Term, there is a Prime in N, whose Index is greater than its Index in A; and consequently cannot measure it, nor can it ever after; because its Index in N is still increasing.

(2°.) If the given Fraction is improper, the Number of fractional Terms in the Series after $\frac{A}{B}$, either increasing or decreasing, is found the same Way as in the last Article. And observe, that if the Number is limited increasing, then it's certainly infinite decreasing; but if it's found infinite, increasing, it may be either finite or infinite, decreasing.

SCHOLIUMS.

1st. In proposing to find how many fractional Terms can be join'd to a given Fraction, we have not distinguish'd betwixt proper and improper Fractions; but that I shall here consider. And,

1°. It's plain, that from an improper Fraction, all the fractional Terms, increasing, must be improper; but decreasing, they may be partly proper and partly improper; and if there stands an Integer below the given improper Fraction, all the intermediate Terms are improper Fractions.

2°. From a given proper Fraction, all the fractional Terms decreasing, are proper; but increasing, they may be partly proper, partly improper; and, in some Cases, all improper.

Hence there are but two Cases concerning the distinct Number of proper and improper Fractions, wherein a Rule is wanted: The first is to determine how many improper Fractions can be joined, decreasing, to a given one in any given Ratio. The second is to find how many proper Fractions can be joined, increasing, to a given one in any Ratio. Now if the Ratio is not Multiple, in either of these Cases, I know of no Rule but actually raising the Series; but if the Ratio is Multiple, they can be solved otherwise: Thus,

Question 1. To find how many improper Fractions can be added, decreasing below a given one, $\frac{A}{B}$, in a given Multiple Ratio, $1 : M$.

Rule. Take the integral Part of the given Number; which Integer call q. Then take the greatest Power of M, which does not exceed q; the Index of that Power is the Number sought.

$\frac{A}{B} : \frac{A}{BM} : \frac{A}{BM^2} : \&c.$

DEMON. The Index of M, expresses the Distance of each Term from $\frac{A}{B}$; and if M^n is the greatest Power of M, not exceeding q, then is $\frac{A}{BM^n}$ the last improper Fraction: For, if any of the Powers of M, in the Divisors, exceeds q, it must be, at least, equal to $q+1$; and as soon as that happens, we have a proper Fraction; because, if $\frac{A}{B}=q$, with a Remainder, therefore qB is less than A, and $\overline{q+1}\times B$, greater than A: So that $\frac{A}{\overline{q+1}\times B}$ is necessarily a proper Fraction, and $\frac{A}{qB}$ an improper: Therefore, if M^n is the greatest Power of M, not exceeding q, then is $\frac{A}{BM^n}$ the last improper Fraction; for a greater Power, as M^{n+1}, must be, at least, equal to $q+1$; and since $\frac{A}{\overline{q+1}\times B}$ is a proper

 Fraction,

Fraction, therefore so is $\frac{A}{BM^{n+1}}$, and consequently $\frac{A}{BM^{n}}$ is the last improper one; and n the Number of Terms sought.

Exam. $\frac{A}{B}=\frac{65}{7}=9\frac{2}{7}$ and M=2, whose third Power, 8, is the greatest, not exceeding 9; therefore 3 is the greatest Number of improper Fractions that can stand in the same Series below $\frac{65}{7}$, in the Ratio 2 : 1, as here $\frac{65}{7}:\frac{65}{14}:\frac{65}{28}:\frac{65}{56}:\frac{65}{112}$.

Question 2. To find how many proper Fractions can be join'd, increasing in the same Series, above a given one, $\frac{A}{B}$, in a given Multiple Ratio 1 : M.

Rule. Take the reciprocal improper Fraction, $\frac{B}{A}$, and find, as in the last Question, how many improper Fractions can be join'd to it, decreasing; and you have the Number sought: For if $\frac{B}{AM^{n}}$ is the last improper Fraction, then taking the whole Series reciprocally, $\frac{AM^{n}}{B}$ is the last proper Fraction after $\frac{A}{B}$, increasing.

2*d*. We may consider the *Problem* unlimited to any particular given Ratio; only in general, suppose the Ratio is Multiple or not Multiple; then,

1°. Suppose an increasing Series such, that it comes at last to an Integer, and a Multiple Ratio; (though there is no particular one given) then find the Prime of B, whose Index is the greatest, that Index less 1 is the Number sought; and if the Product of that Prime, multiplied into all the other Primes of B, is made the Consequent of a Multiple Ratio, by that you may find an Example of the Series: The Reason of the Rule is evident. But again,

2°. If the Problem is limited to a Ratio not Multiple, the Series increasing as before; then take the Prime of B, which has the greatest Index, provided also that there be a Prime in A, which has an equal or greater Index; the Index of that Prime in B, less 1, is the Number sought: And you may find an Example of the Series by a Ratio whose Antecedent is that Prime of A, and its Consequent the Product of all the Primes of B; provided it be a Number greater than the Antecedent; else, take some Multiple of it greater than the Antecedent. Hence, if the Index of the Prime, referred to in B, is 1, there can be no improper Fraction joined with the given Number.

Example for both these Articles. $\frac{A}{B}=\frac{405}{56}=\frac{3\times3\times3\times3\times5}{2\times2\times2\times7}$. Here 2 is the Prime in B, whose Index, 3, is the greatest; and therefore 2 is the Number sought, in the first Article, and 1 : 14 is a Ratio agreeing to this Solution.

Again, If the Ratio is supposed not Multiple, 2 is also the Number sought; because, according to the Rule, there is a Prime in A, whose Index is greater than 3: And if you take for a Ratio, 3 : 14, it will agree to the Solution.

But suppose $\frac{A}{B}=\frac{105}{26}=\frac{3\times5\times7}{2\times13}$; if there comes an Integer into the Series, it must be the very first Term after $\frac{A}{B}$; because there is no Index either in A or B, greater than 1; for no Ratio Multiple can answer the Supposition of the Series coming at last to an Integer, except this, 1 : 26 (=2×13) and it's plain, that the very first Term

Term added will be 105. Again, No Ratio, not Multiple, can answer, but one of these in which the Antecedent is, 3. or 5. or 7. or 3×5. or 3×7. or 5×7. or $3\times5\times7$. and the Consequent 2×13, or some Multiple of this that is greater than the Antecedent: In all which Cases it's manifest, that the first Term added will be an Integer.

3°. Suppose a given improper Fraction, and that a Series ought to proceed from it, decreasing, in a Multiple Ratio: To find the greatest Number of fractional Terms, improper, that can be join'd: From a Table of Powers seek the greatest Power (*i. e.* a Power of the greatest Index) which is a Number not greater than q, the integral Part of $\frac{A}{B}$; and of whose Root the next greater Power is greater than q; the Index of the former Power is the Number sought; and that Root is the Antecedent of a Multiple Ratio, which will make a Series, having the Number of fractional Terms found, after $\frac{A}{B}$.

Exam. $\frac{A}{B}=\frac{65}{7}=9\frac{2}{7}$. The Number sought is 3; for 8, the third Power of 2, is the greatest Power, which does not exceed 9; and the next Power of whose Root (*viz.* 16.) does exceed 9: And by the Ratio, 2 : 1, we have the Series, $\frac{65}{7}:\frac{65}{14}:\frac{65}{28}:\frac{65}{56}$.

PROBLEM XII.

A Fraction, $\frac{A}{B}$, being given for the lesser Extreme of a Series, to find a Ratio, in which a given Number, n, of fractional Terms, and no more, can possibly be in the same Series above the given one.

SOLUTION. 1°. Suppose we are to find a Multiple Ratio; then find the Prime that's oftnest involved in B; if its Index does not exceed n, then the Problem is impossible: For M, the Consequent of any Ratio, 1 : M, which will bring the Series to a whole Number, must contain all the Primes of B; therefore it contains the Root, at least, of that Prime which is most involved in B; but unless the Index of that Involution be greater than n, M^n will be equal to, or greater than B; and consequently the nth Term after the given Extreme, is a whole Number; or, perhaps, there may be one before this. Again, though the Index of some of the Primes of B exceeds n, yet the Problem may be impossible; for to make it possible, the Index of some of these Primes must be such, that there be some inferior Power, whose Index, multiplied by n, makes a Product less than its Index in B; but multiplied by $n+1$, makes a Number not less than that Index; and then the Product of that inferior Power of any such Prime in B, multiplied continually into such Powers of all the other Primes of B, as that their Indexes, multiplied by $n+1$, makes a Number not less than their Indexes in B, may be taken for the Consequent of the Ratio sought. The Reason of all this is manifest, by considering that an Integer can never come into the Series, till the Power of M, in the Numerator, contains the several Primes of B, involved, at least, as oft: And as soon as that happens, we have a whole Number.

$$\frac{A}{B}:\frac{AM}{B}:\frac{AM^2}{B}:\frac{AM^3}{B}:\&c.$$

Exam. 1. $\frac{A}{B}=\frac{36}{175}=\frac{2\times2\times3\times3}{5\times5\times7}$, and $n=3$. The Problem is impossible; because there is no Index of a Prime in B, which exceeds 3.

Exam.

Exam. 2. $\frac{A}{B}=\frac{25}{3456}=\frac{5\times5}{3\times3\times3\times2\times2\times2\times2\times2\times2\times2}$, and $n=3$; then is the Problem possible; because the Index of 2 in B, is 7; and if we take the second Power of 2 (*viz.* 4) the Index of this, multiplied by 3, is 6; and multiplied by 3+1, or 4, makes 8; therefore the Ratio sought is 1 : 12; and the Series is $\frac{25}{3456}:\frac{300}{3456}:\frac{3600}{3456}:\frac{43200}{3456}:\frac{518400}{3456}=150$. And the Reason of the Rule will appear in this Example, by setting it down in this Manner, *viz.*

$$\frac{25}{3^2\times2^7}:\frac{25\times3\times2^2}{3^3\times2^7}:\frac{25\times3^2\times2^4}{3^3\times2^7}:\frac{25\times3^3\times2^6}{3^3\times2^7}:\frac{25\times3^4\times2^8}{3^3\times2^7}=25\times3\times2.$$

Case II. *Suppose a Ratio not Multiple, as* N : M, *is to be found.*

The Problem is possible only when there is some Prime in B, whose Index is greater than n, in the Manner described in the preceding Case; and when at the same Time there is some Prime in A, whose Index is greater than n, so much that there be some inferior Power of it, whose Index multiplied by $n+1$, shall produce a Number not exceeding its Index in A. Then for the Consequent of the Ratio, take such a Number as was directed in the first Case; and for the Antecedent, take any one, or the Product of any two or more of such inferior Powers of the Primes in A, whose Indexes are as above described. And observe, that if the Consequent, found by the Direction of the first Case, is a Number less than any of the Antecedents, found by the Direction of this Rule, then multiply it by some other Prime Number, which is not in B; so that the Product be greater than the Antecedent.

The *Reason* of all this Rule is to be found from the same Consideration as that of the last Case.

Exam. $\frac{A}{B}=\frac{135}{56}=\frac{5\times3\times3\times3}{7\times2\times2\times2}$, and $n=2$. The Problem is possible; because the Index of 2 in B, is 3, and $2\times1=2$ (1 being the Index of the first Power of 2) and 2+1, or $3\times1=3$. Then again, the Index of 3 in A is 3, and $3\times1=3$ (1 being the Index of the first Power of 3) and for the Ratio sought, we have 3 : 14 ($=2\times7$) as in this Series, $\frac{135}{56}:\frac{1890}{168}:\frac{26460}{504}:\frac{370440}{1512}=245$; which we may also represent in the following Manner; whereby the Reason of the Rule will clearly appear,

$$\frac{5\times3^3}{7\times2^3}:\frac{5\times3^3\times2\times7}{7\times2^3\times3}:\frac{5\times3^3\times2^2\times7^2}{7\times2^3\times3^2}:\frac{5\times3^3\times2^3\times7^3}{7\times2^3\times3^3}=5\times7^2.$$

SCHOLIUMS.

1*st*. If it's proposed to find a Ratio, in which no Fractional Term can be added; take any Number, which is a Multiple of B, for the Consequent, and any *aliquot* Part of A, or A it self, for the Antecedent. The Reason is plain.

Or, if it's proposed that all the Terms be Fractional *ad Infinitum*; we have no more to do, but take $\frac{M}{N}$, such, that N contain some Prime which is not in A, or the same Prime oftner involved; or take M, such, that it contain not all the different Primes of B.

2*d*. The greatest limited Number of Fractional Terms that can possibly be joined in the same Series with a Fraction, must necessarily lie all on one Side; and not partly above and partly below; because Fractions cannot lie betwixt Integers in the same Series: And as the greatest Number may lie either above or below, in different Cases; the Reason why I have limited the Problem to an increasing Series, is plainly because I know of no determinate Rule that will reach to all Cases for a decreasing Series: The Reason of which seems to be this, That it does not depend only upon the Consideration of like or unlike Primes, and their different Involutions, but also upon their particular Quantities and Proportions; yet we have these two Particulars to observe:

(1°.) That as from a proper Fraction a Series decreases in proper Fractions, for ever, in whatever Ratio: So,

(2.) The given Fraction being improper, if the Ratio is required to be Multiple, the Problem is impossible; the Reason of which has been explained in *Probl.* XI. *Case* I. But if it's only required to find such a Multiple Ratio, in which a given Number of improper Fractions, and no more, can be joined to a given improper Fraction, $\frac{A}{B}$, decreasing; this we can find, *thus*, Take q, the integral Part of the given mix'd Number, and seek a prime Number, M, different from any of the Primes of A; and such also, that it's n Power (n being the given Number of Terms) be the greatest of its Powers, which does not exceed q; and that Prime is the Antecedent of a Ratio, which solves the Question: For it has been shewn, that $\frac{A}{BM^n}$ is the last improper Fraction; and that it is not an Integer, also that none of the added Terms are so is plain; because M is a Prime different from any of these in A.

PROBLEM XIII.

To find a Fraction with which may be joined in the same Series, increasing, a given Number, and no more fractional Terms, in a given Ratio.

SOLUTION.

Case I. *Suppose a Ratio Multiple*, 1 : M, *and the given Number of Terms*, n.

$$\frac{A}{B} : \frac{AM}{B} : \frac{AM^2}{B} : \&c.$$

Resolve M into all its Primes; and for the Denominator sought, take any Number, B, composed of some or all of the same Primes, and no other than are in M: But let some one of them, at least, be oftner involved in B than in M; so much, that taking all of these, whose Indexes in B and M have the same Difference, and that too, the greatest of any other of them; and chusing any one of these, whose Index is the greatest; the Product of its Index in M, multiplied by n, shall be less than its Index in B; but multiplied by $n+1$, it shall not be less.

Then for a Numerator, take A, any Number whatever Prime to B; and you have the Fraction sought; which is proper or improper, as A is less or greater than B.

Exam. M$=72$, and $n=4$. The Primes of M are $2\times2\times2\times3\times3$: The Index of 3 in 72 is 2; which multiplied by 4 gives 8; and multiplied by $4+1$, or 5, gives 10: Therefore I take a Number, for B, which has 3 involved in it to the 9th Power; because 9 is greater than 8, but not greater than 10: Such a Number is the 9th Power of 3 it self, *viz.* 19683; which we may also multiply by any other of the Primes of M, as 2, or any such Power of any of these Primes, whose Index is either not less than its Index in M, or which does not want of it the Difference betwixt the Indexes of the first assumed Primes in B and M. The rest of the Work is obvious.

Case

Case II. *Suppose the Ratio not Multiple, as* N : M.

Find the Denominator, B, the same Way as before, from the Primes of M: Then for a Numerator, take A, any Number composed of all the Primes of N (and any other you please, provided they are none of those of B) but let them be so much oftner involved, that taking all of these whose Indexes in N and A have the same Difference, and that, the least of any other of them; and chusing that one of them whose Index is greatest, the Product of its Index in N by $n+1$, shall not exceed its Index in A.

The Invention of the Numbers sought, in these Cases, has no Difficulty; and the *Reason* is contained in what has been explained in the preceding Problems, especially *Probl.* XI. And 'tis to be observed too, that this Problem is always possible, *i. e.* with any Ratio, and any Number of Terms.

Scholiums.

1st. Suppose the Number required ought to be such, that the very first Term after it is a whole Number; then for a Multiple Ratio, 1 : M, take for B any *aliquot* Part of M, or M it self; and for A, any Number prime to B. Again, for a Ratio not Multiple, as N : M, take B, as before; and for A take N, or any Multiple of it, so that $\frac{A}{B}$ be not a whole Number.

If the Number of Terms to be added is infinite, take B, such, that it have some Prime, which is not in M.

2d. If no particular Ratio is given, *i. e.* If it's required to find such a Fraction, that a given Number is the greatest Number of Fractional Terms that can possibly be joined with it in the same Series increasing, in any Ratio whatever; then the Problem is impossible: For no such Fraction can be found; because with every Multiple Ratio, an infinite Number of Fractional Terms can be join'd in the same Series with any Fraction; and it may be so too, with many Ratios not Multiple: But if we suppose the Series limited to this Condition, that it shall not continue for ever in Fractions; then,

(1°.) Suppose a Multiple Ratio: Take for B, any Number whereof that Prime which is oftnest involved shall have $n+1$ for its Index; and for A, take any Number Prime to B: And to find an Example of the Series, take for M, (the Consequent of the Ratio) the Product of the Roots of all the Primes in B, or that Product multiplied by any other Prime.

(2°.) Suppose a Ratio not Multiple; take B, as before; and for A, let it be a Number prime to B, and such that all its Primes be involved to the $n+1$ Power: Then the Invention of a Ratio, answering the Problem, is this; Take the Consequent M, as before; and for the Antecedent N, take any of the Primes of A, or the Product of any two or more of them, provided it be less than the Consequent.

3d. What has been said in *Schol.* 2*d*, of the last Problem, may be repeated here, for the Reason why this Problem is limited to an increasing Series: Yet concerning decreasing Series, we have to add these Particulars, *viz.*

(1°.) Suppose the given Ratio is Multiple; then for a decreasing Series the Problem is impossible: For the Number of Fractional Terms will be infinite after any Fraction.

(2°.) Suppose we ought to find a mix'd Number, with which may be join'd a given Number, n, of mix'd Terms, and no more, decreasing in a given Multiple Ratio, M : 1; it may be found, *thus*; Take the n Power of M, and then any Number, q, not exceeding M^n. *Lastly*, Take any mix'd Number, $\frac{A}{B}$, whose integral Part is q;

but take Care also that A be such that there be some Prime in M, which is not in A, or the same Prime oftner involved.

(3°.) If the mix'd Number in the last Article is required to be invented in its least Terms; take for B any Number at Pleasure, which is not a Multiple of 2 (*i. e.* any odd Number) and for A, multiply B by the n Power of 2; to the Product add 1 [or more generally add any Number, whereby the Sum may be Prime to B; and at the same Time a Number lesser than the Product of B by the $n+1$ Power of 2] and make this Sum the Numerator: And if 2 is made the Ratio, 'tis evident there will be n mix'd Terms and no more. Again, with a greater Ratio than 2, it's certain there will be fewer mix'd Terms; for the Denominator will sooner exceed the Numerator: Hence 'tis plain that $\frac{A}{B}$ is such a mix'd Number as was required. Or see the Reason of it, *thus*; The Number found, according to this Rule, is thus expressed, $\frac{B\times 2^n+1}{B}$; to which n Number of Terms being joined in the Ratio, 2 : 1, the last of them is $\frac{B\times 2^n+1}{B\times 2^n}$, which is an improper Fraction; and the very next Term would be a proper one, *viz.* $\frac{B\times 2^n+1}{B\times 2^{n+1}}$; where 2^{n+1} is manifestly greater than 2^n+1.

§. IV. *Of Numbers* odd *and* even.

Theorems I, II, III, IV.

In *Addition of Numbers* these Things are true;

1*st*. The Sum of two or more even Numbers, is an even Number; for 2 measures each of the Parts, therefore it measures the Whole (*Ax.* 1.) *Exam.* $4+6+8=18$.

2*d*. The Sum of two odd Numbers is an even Number; for 1 taken from each of them, leaves an even Number, but $1+1=2$: So that the Sum is, the Sum of two even Numbers and 2 added. *Exam.* $5+7=12=4+6+2$.

3*d*. The Sum of an even and an odd Number, is an odd Number; for 2 cannot measure both the Parts, since it cannot measure an odd Number; (*Cor.* 1. *Ax.* 3.) therefore it cannot measure the Whole (*Ax.* 1.) Or thus, 1 taken from the odd Number, leaves an even; so that the Sum is the Sum of 2 even Numbers, (which is an even) and 1, which makes an odd Number (by *Cor.* 4. *Defin.* 8.)

4*th*. If more than 2 Numbers are added, which are all odd Numbers, or partly even partly odd; the Sum is even or odd, according as the Number of odd Parts is even or odd; which follows easily from the former Articles.

Exam. $3+5+7=15$, and $3+5+13+19=40$. Also,
$3+4+6=13$, and $3+5+4=12$.

Theorems V, VI, VII.

In *Substraction* these Truths are evident, being the Reverse of the former, *viz.*

5*th*. The Difference of 2 even Numbers, is an even Number: So $8-4=4$.

6*th*. The Difference of 2 odd Numbers, is an even Number: So $9-5=4$.

7*th*. The Difference of an even and an odd Number, is an odd Number: So $14-5=9$, and $19-8=11$.

Theorems VIII, IX.

In *Multiplication* these Truths are evident from the first *Theorem*; because Multiplication is only a repeated Addition.

8th. The Product of two even Numbers, or of one odd and even Number, is an even Number; for it's only the Repetition of an even Number, or a Sum of even Numbers: So $4\times8=32$, and $6\times7=42$.

9th. Two odd Numbers produce an odd Number; for it's the Sum of an odd Number of odd Parts: So $3\times7=21$.

Coroll. The Powers of an even Number are all even; and of an odd Number are all odd. Also the Product of more than 2 odd or even Numbers, is odd or even; and the Product of several Factors partly odd, partly even, is even.

Theorems X, XI, XII.

In *Division*, these Truths follow from the last; because this is the Reverse of that.

10th. An even Number measures an even Number by an even, or an odd, in different Cases: So $12\div4=3$, and $12\div6=2$; the Reverse of which is, that the Measure of an even Number may be odd or even.

11th. An odd Number measures an even, only by an even: So $24\div3=8$.

12th. An odd Number measures an odd, only by an odd: So $35\div7=5$.

COROLLARIES.

1st. The Roots of odd or even Numbers are all odd or even.

2d. The Number which an odd Number measures, may be either odd or even; which is plain from the second and third Articles: But the Number which an even Number measures, must be even; else the Product of two even Numbers, or of an even and odd (*viz.* the Quote and Divisor) would be odd, contrary to *Theor.* 8; which last Part we may also express thus, *viz.* An even Number cannot measure an odd: Or also thus, There is no even Number in the Composition of an odd; and so, lastly, An odd Number only can measure an odd.

Theorem XIII.

There is no Number whatever (excluding 1) that will measure all odd Numbers, because an infinite Number of those are prime Numbers; but all even Numbers have a common Measure, *viz.* 2, from the Definition.

Theorem XIV.

Two odd Numbers, that differ by 2 (*i. e.* every two adjacent Terms in the Series of odd Numbers) are Incommensurable; for dividing them by the Rule for finding their greatest common Measure, the first Remainder is 2; and the next will be 1; which is the greatest common Measure.

Theorem XV.

If an odd Number measures an even, it will also measure the Half of the even Number. *Exam.* 3 measures both 12 and 6.

A)	B	(C
3	12	4

Demon. If A, an odd Number, measures B, an even, let the Quote be C, it is an even Number (*Theor.* XI.) which will back again measure B by A; but B and C, being both even, are measurable by 2. Also $C : B :: \frac{C}{2} : \frac{B}{2}$. Now since C measures B by A, therefore $\frac{C}{2}$ measures $\frac{B}{2}$ by A; and back again, A measures $\frac{B}{2}$ by $\frac{C}{2}$: Otherwise thus, any even Number may be expressed 2N; and if A measures 2N by C; then is $AC=2N$, and $A : N :: 2 : c$. But 2 measures C, which is an even Number; therefore A measures N, the Half of 2N.

THEOREM XVI.

If an odd Number, A, be Incommensurable with any Number, B; it will be so also with the Double of B, or 2B.

m | B, A . d, 2B	

DEMON. If A and 2B are Commensurable, suppose m their common Measure; it must be an odd Number, because it measures an odd Number (for no even can measure an odd, *Cor.* 2. *Theor.* XII.) let the Quote be d, it is also an odd Number (*Theor.* XII.) Again, 2B is an even Number; and because m, an odd Number, measures it, therefore it measures its Half, B (*Theor.* XV.) consequently, A, B, are not Incommensurable, contrary to Supposition.

COROLL. For the same Reason, A will be Incommensurable with any Product of B, multiplied into some odd Number.

THEOREM XVII.

All even Numbers are either some Power of 2, or some of those multiplied into some odd Number.

2) A
2) B
2) C
2) D
E

DEMON. All the Powers of 2 are even Numbers (*Cor.* after *Theor.* IX.) but an even Number, which is not any Power of 2, is the Product of such a Power by some odd Number: Suppose any even Number, A, divide it by 2, and let the Quote be B; divide this again by 2, and let the Quote be C; and so on, as in the Margin: As long as 2 measures the succeeding Quotes, the last Quote will be either 1, or some odd Number; for as long as any of those Quotes is an even Number, it's again measurable by 2; and so that is not the last Quote. If it's 1, then A is some Power of 2; for it's the Product of all these Divisors; and if it's not 1, it must therefore be some odd Number; consequently A is composed of those Factors, *viz.* that odd Number which is the last Quote, and 2 as oft involved as the Number of Divisions, *i. e.* such a Power of 2, whose Index is the Number of Divisions: Thus, if B is an odd Number, then is $A=2B$; if C is an odd Number, $A=2\times2\times C$, and so on.

Or, the whole Demonstration may be made thus; No even Number is a Prime but 2; and all other Primes are odd, by Definitions; therefore, let any even Number, A, be reduced to its component Primes, some one, or more of those, must be 2; else it cannot be an even Number: Then either there is no other Prime amongst these but the Number 2; so that A is a Power of 2; or if there are also other Primes, they must be odd Numbers; and if there are more than one such, their Product is an odd Number (*Theor.* IX.) Consequently A is the Product of some Power of 2, by some odd Number.

THEOREM XVIII.

All even Numbers, above 2, have some one of these Qualities, *viz.* they are either,

1*st*. Evenly even only, *i. e.* They are not also oddly even; and such are all the superior Powers of 2, and none other.

2*d*. Oddly even only, *i. e.* They are not also evenly even; and such are all the Doubles of every odd Number, or the Products by 2, and none other.

3*d*. Both evenly even, and oddly even; and such are the Products of any odd Number, by any Power of 2 (above 2) and none other.

DEMON. It's evident that all even Numbers, above 2, must have some one of the three Qualities mentioned; and that the same Number can have but one of them. What is to be demonstrated, is therefore this; That the several Classes of Numbers describ'd, have the Qualities assigned to them; and that none other have these Qualities.

1°. The

1°. The superior Powers of 2 are evenly even; this is manifest: Then again, they are not oddly even; for 2 being a Prime, no Number can measure any of its Powers, but some other of its Powers (*Cor. 6. Theor.* X. *Ch.* 1.) which are all even Numbers; therefore no odd Number can measure them, *i. e.* they are not oddly even.

2°. The Multiple of any odd Number A by 2, or 2A, is odly even by the Definition; but it's not evenly even; for if we suppose it is, let $2A = BC$ (two even Numbers) then $2 : B :: C : A$, but 2 measures the even Number, B; and therefore so must the even Number, C, measure the odd Number, A; which is impossible (*Theor.* XII. *Cor.* 2.)

3°. Any superior Power of 2, multiplied into an odd Number, produces a Number which is both evenly even and oddly even: It is oddly even by the Definition; but it's also evenly even; for let the odd Number concerned in it, be called o, and the supposed Power of 2 be m; then is m equal to the next Power below, multiplied by 2: Call that mix'd Power, p, so is $2p = m$, consequently $o \times m = 2 \times p \times o$; but po is an even Number (*viz.* the Multiple of an even Number, p) therefore $2 \times po$ ($= mo$) is evenly even.

4°. There can be no even Number of any of the Qualities mentioned, but these described; for it's shewn, that none of the several Classes can be of any other of these Qualities, but that assigned to it; and these three Classes comprehend all even Numbers, by (*Theor.* XVII.) Since they are either Powers of 2, or some of those multiplied by some odd Number; the first Class comprehends all the superior Powers of 2; the second comprehends 2 and its Products by all odd Numbers; the third comprehends the Product of all the superior Powers of 2 by all odd Numbers.

From the Nature and Manner of producing the Species of even Numbers, here explained, the following Consequences will easily appear.

COROLLARIES.

1*st*. Every Number, which is evenly even, has an even Half, or is measurable by 4; for it's either some superior Power of 2, as 4 . 8 . 16 . &c. each of which is measurable by 4; or it's the Multiple of such a Power by some odd Number.

2*d*. The Product of two Numbers, evenly even only, is evenly even only; being the Product of two Powers of 2; which is also some Power of 2 from the Nature of Powers.

3*d*. A Number evenly even only, multiplied by any Number oddly even only, or both oddly and evenly even, produces a Number which is both oddly and evenly even; because there is in the Composition of the Product some odd Number, and also some higher Power of 2.

4*th*. A Number oddly even only, multiplied by any even Number whatever, produces a Number both oddly and evenly even; but multiplied by an odd Number, produces a Number oddly even only; the Reason of both which is manifest.

5*th*. A Number both oddly and evenly even, multiplied by any Number whatever, produces a Number both oddly and evenly even.

6*th*. An evenly even only, can be measured by none but another such, or the Root 2; because it is the Power of a Prime Number (2) which can be measured by none but some of the inferior Powers of that Prime; wherefore an evenly even only, being measured by another such, quotes another such, or 2.

7*th*. A Number both oddly and evenly even, may be measured by any Kind of Number whatever.

8*th*. A Number oddly even only, can be measured by no Number, but 2 or some odd Number: Whence again,

9*th*. The

9*th*. The Number which is meaſured by an evenly even only, is either evenly even only, or both oddly and evenly even; and the Number which one both oddly and evenly even, or one oddly even only, meaſures, is alſo both oddly and evenly even.

10*th*. If you take the Series of even Numbers, 2. 4. 6. 8. 10. 12. 14 *&c.* and beginning at 6, take every other Term, *i. e.* paſſing one take the next, *viz.* 6. 10. 14 *&c.* you have the Series of Numbers oddly even only, becauſe the Series of even Numbers are the Doubles of the ſeveral Terms of the natural Progreſſion 1, 2, 3 *&c.* whereof every other Term, beginning at 3, make the Series of odd Numbers, and the Doubles of thoſe is the Series of Numbers oddly even only; ſo that 6 being 2×3, every other Term after this makes the Series oddly even only.

THEOREM XIX.

1	3.	5.	7.	9.	11, *&c.*
2	6.	10.	14.	18.	22, *&c.*
4	12.	20.	28.	36.	44, *&c.*
8	24.	40.	56.	72.	88, *&c.*
16	48.	80.	112.	144.	176, *&c.*

Take the Series of odd Numbers from 1, then the Series of their Doubles, then the Doubles of this Series, and ſo on; as the firſt Line is the Progreſſion of odd Numbers, ſo the ſecond Line is the Series of Numbers oddly even only, and the firſt Column on the left Hand (excluding 1.2) is the Series of Numbers evenly even only: The other Columns below the ſecond Line containing the Numbers that are both oddly and evenly even.

The Truth of all this is evident from the Definitions of theſe Kind of Numbers, and the Conſtruction of this Table; upon which this alſo is remarkable, *viz.* That each Line (taken from left to right) is an Arithmetical Progreſſion, whoſe common Difference is double the firſt Term; the Reaſon of which will be plain from theſe Conſiderations, *viz.* 1°. Becauſe it is ſo in the firſt Line or Series of odd Numbers. 2°. By the Conſtruction of the Table the firſt Term of every Line is double the firſt Term of the preceding. 3°. The Product of any Arithmetical Progreſſion is an Arithmetical Progreſſion, whoſe Difference is the Product of the former Difference by the common Multiplier. Again, each Column from top to bottom is a Geometrical Progreſſion in the Ratio of 1 : 2, as is plain from the Conſtruction. So that the whole Syſtem of Numbers that are both oddly and evenly even, proceed either from the ſeveral Numbers which (excluding 2) are oddly even only, by multiplying each of theſe continually by 2, making ſo many Geometrical Progreſſions; or from the Numbers evenly even only, by adding to theſe continually their Doubles, making ſo many Arithmetical Progreſſions.

THEOREM XX.

The Sum of Numbers that are all oddly even only, may be oddly even only, or evenly even only, or both; *Thus*, if the Number of Terms added is odd, the Sum is oddly even only; but if it's even, the Sum is evenly even only, or both oddly and evenly even.

Exam. $6+10+14=30$, oddly even only,
$6+10 = 16$, evenly even only,
$6+10+14+18=48$, both oddly and evenly even.

DEMON. Let a, b, c, d, &c. be any odd Numbers, then are $2a$, $2b$, $2c$, $2d$, &c. Numbers oddly even only; and their Sum $2a, +2b+2c, + 2d$ &c.$=2\times\overline{a+b+c+d}$ &c. is oddly even only, if the Number of Terms added is odd; for an odd Number of odd Numbers makes together an odd Number, and an odd Number doubled makes a Number oddly even only. Again, if the Number of Terms added is even, then their Sum is even, and may be repreſented by 2 A, and therefore its double is 4A, or 2×2A, which, it's plain, cannot be a Number oddly even only; and therefore muſt be either

either evenly even only, or both oddly even and evenly even. That in some Cases it will be the one way, in some the other, the preceding Examples shew; and you'll learn afterwards how to invent Examples of each Kind.

THEOREM XXI.

The Sum of any Number of Numbers that are all evenly even only, is both oddly and evenly even. *Exam.* $4+8+16+32=60=2\times30=6\times10$.

DEMON. The given Numbers may be represented thus, 2^n, 2^{n+r}, 2^{n+s} 2^{n+t}; and in the first Place, because $2^{n+r}=2^n\times2^r$, therefore $2^n+2^{n+r}=2^n+2^n\times2^r=2^n\times\overline{2^r+1}$. But 2^r+1 is odd, therefore $2^n\times\overline{2^r+1}$ is oddly and evenly even. This Sum we may now represent by $2^n\times o$ (o being the odd Number $=2^r+1$) then is $2^n\times o+2^{n+s}=2^n\times\overline{2^s+o}$; which for the same Reason as before is both oddly and evenly even. Call this Sum again $2^n\times o$, and the next Sum is $2^n\times o+2^{n+t}=2^n\times\overline{2^t+o}$, which is both oddly and evenly even; and so on.

THEOREM XXII.

The Sum of any Number of Terms, all both oddly and evenly even, is either evenly even only, or both oddly and evenly even; and particularly, if the Number of Terms is 2, the Sum is both oddly and evenly even.

DEMON. The Sum cannot be oddly even only, because each of the Terms has an even half, or is measurable by 4; whence the Sum is also measurable by 4, and consequently it is either evenly even only, or both oddly and evenly even (of which you'll find Examples and Rules how to invent them afterwards). Again, if there are but 2 Terms, the Sum is both oddly and evenly even: For every Number, both oddly and evenly even, is the Product of an odd Number, by some superior Power of 2; wherefore let a, o represent two odd Numbers, and 2^n, 2^{n+r}, two Powers of 2, then will $2^n\times o$, $2^{n+r}\times a$, represent any two Numbers both oddly and evenly even; but their Sum is $2^n\times o+2^{n+r}\times a=2^n\times\overline{o+2^r\times a}$; and 2^r being even, $2^r\times a$ is even also, and $o+2^r\times a$ is odd; consequently $\overline{2^n\times o+2^r\times a}$ (or $2^n\times o+2^{n+r}\times a$) is both oddly and evenly even, being the Product of an odd Number by a superior Power of 2.

THEOREM XXIII.

The Sum of any Number of Terms all evenly even only, and any Number all both oddly and evenly even is either evenly even only, or both oddly and evenly even; and particularly, one Number evenly even only, being added to another both oddly and evenly even, the Sum is both oddly and evenly even.

DEMON. The Sum cannot be oddly even only, because each Part is measurable by 4, therefore it is either evenly even only, or both oddly and evenly even. Again, if there is but one of each, the Sum is both oddly and evenly even; for the 2 supposed Numbers may be expressed thus, 2^n, and $2^{n+r}\times o$; whose Sum is $=2^n+2^{n+r}\times o=2^n\times\overline{1+2^r\times o}$; but 2^r is even, and o is odd, therefore $2^r\times o$ is even, and therefore $1+2^r\times o$ is odd, and hence $2^n\times\overline{1+2^r\times o}$ is both oddly and evenly even; or the 2 Numbers may be represented thus, $\overline{2^{n+r}}$; $2^n\times o$, whose Sum is $2^{n+r}+2^n\times o=2^n\times\overline{2^r+o}$; but 2^r+o is odd, and 2^n even, hence $2^n\times\overline{2^r+o}$ is both oddly and evenly even.

THEOREM XXIV.

Any Number of Terms oddly even only, with any Number evenly even only, or both oddly and evenly even, make a Sum either oddly even only, or both oddly and evenly even; particularly, 1°. Any odd Number (among which reckon 1) of Terms,

 all

all of them oddly even only, added to one or any Number, all evenly even only, or both oddly and evenly even, makes the Sum oddly even only.

2°. Any even Number of Terms all oddly even only, added to one or any Number of Terms all evenly even only, or both oddly and evenly even, make a Sum both oddly and evenly even.

Demon. For Article first, which must be subdivided thus,

(1°.) Suppose one Number, oddly even only, added to one either evenly even only, or both oddly and evenly even, their Sum is oddly even only; for let o be any odd Number, and e even, then $2\times o$ represents any Number oddly even only; and if e is a Power of 2, then $2\times e$ represents any Number evenly even only; else $2\times e$ is a Number both oddly and evenly even; but it's plain, that $2\times o+2\times e=2\times\overline{o+e}$, and $o+e$ is odd, hence $2\times o+2\times e$ is oddly even only.

(2°.) Suppose more generally any odd Number (including 1) of Terms all oddly even only, added to any Number, evenly even only, or both oddly and evenly even, the total Sum is oddly even only; for the Sum of the Numbers oddly even only is oddly even only (*Theor.* XX.) and the Sum of these that are evenly even only is both oddly and evenly even (*Theor.* XXI.) which two Sums make the total Sum oddly even only (by what's last demonstrated.) But if the other Part consists of Numbers, both oddly and evenly even, then their Sum is either evenly even only, or both oddly and evenly even (*Theor.* XXII.) either of which added to the former, which is oddly even only, the Total is oddly even only (*Case* I.)

For Article second, the Sum of an even Number of Terms all oddly even, is either evenly even only, or both oddly and evenly even (*Theor.* XX.) then the Sum of any Number of Terms, all evenly even only, is both oddly and evenly even (*Theor.* XXI.) Also the Sum of any Number of Terms both oddly and evenly even, is either evenly even only, or both oddly and evenly even (*Theor.* XXII.) wherefore it's plain, that what we have to consider in this Article is this, *viz.* What kind of a Sum is that of 2 Numbers, both of them evenly even only, or both of them oddly and also evenly even; or the one evenly even only, and the other both oddly and evenly even (for of these Kinds are the Sums of the two Classes of Numbers added) in all which three Cases the Sum is both oddly and evenly even, by *Theor.* XXI, XXII, and XXIII, the last part of which shews the Truth of the last Case.

Theorem XXV.

If there are three Numbers in Arithmetical Progression, whereof the middle Term is evenly even only, and one of the Extremes oddly even only, the other Extreme is also oddly even only.

Demon. Let the three Terms be $2\times o$, 2^n, N, the first being oddly even only (for o is an odd Number) and the second being evenly even only, or some Power of 2, then is $2\times o+N=2\times 2^n=2^{n+1}$; but since 2^{n+1} is even, so must $2\times o+N$; and because also $2\times o$ is even, so is the Remainder N. Let it be supposed that $N=2a$, then is $2^{n+1}=2\times o+2a=2\times\overline{o+a}$, but a is an odd Number, for else $\overline{o+a}$ will be odd (*viz.* the Sum of an odd Number, and even Number) and then $2\times\overline{o+a}$ is oddly even only, *i. e.* 2^{n+1} $(=2\times\overline{o+a})$ is oddly even only, which is impossible, for 2^{n+1} is a Power of 2, or evenly even only; wherefore a must be odd, and consequently N $(=2a)$ is oddly even only.

PROBLEM I.

To find a proposed even Number of Numbers, which are all oddly even only, and whose Sum is evenly even only.

Rule

Rule 1. If the proposed Number is 2, take any Number oddly even only (as A in *Exam.* 1st.) Also any Number evenly even only and which is greater than the former Number (as B) then take a third in Arithmetical Progession to A, B, as, C; and A, C, are the Numbers sought.

Exam. 1.		
A,	B,	C,
14,	16,	18,

Demon. By *Theor.* XXV, C is oddly even only; then $A+C=2B$, which is a Number evenly even only, *viz.* some Power of 2, because B is such.

2°. If the Number is 4, take 2 Numbers oddly even only, as A, B, *Example* second; also any Power of 2, as C, which is a greater Number than B; then take D, E, as much greater than C, as B A are lesser; and A, B, D, E, are the Numbers sought.

Exam. 2.				
A :	B :	C :	D :	E
6 :	10 :	16 :	22 :	26

Demon. D, E, are oddly even only, by *Theor.* XXV. and $B+D=2C$, also $A+E=2C$, therefore $A+B+D+E=4C$, which is a Power of 2, because both 4 and C are so.

3°. Let the proposed Number be any even Number, above 4; find first four of the Numbers sought, as in the last Case; then take the next Power of 2 above C, as G; and below it take a Number, as F, oddly even only, and which is different from any of the preceding, and another as far above it as H; then take in the next Power of 2, as K, and take below it a Number oddly even only, and another as far above; and so on, till you have as many Numbers as are required.

A :	B :	C :	D :	E :	F	G	H	I	K	L
6 :	10	:	22 :	26 :	30	:	34 :	62	:	66
		16				32			64	

Demon. For the first 4 we have the Demonstration already: Then for the next 2; $F+H=2G$; But $2C=G$, therefore $4C\ (=A+B+D+E)$ is $=2G$, consequently $A+B+D+E+F+H=4G$, which is a Power of 2. The Reason goes on the same Way to the next Two, and so for ever.

Schol. If the proposed Number is it self some Power of 2, we may work thus; take any Number of Terms all oddly even only, which is equal to the Half of the Number proposed, then take a Power of 2 greater than the greatest of them, and as many Terms above it, at the same Distance as the former half are below it; thus, to find 8 Terms, I first take 4, as A, B, C, D, then a Power of 2, as E, greater than

A,	B,	C,	D,	E,	F,	G,	H,	I
6 ·	10 ·	14 ·	18 ·	32 ·	46 ·	50 ·	54 ·	58] ·

D; and lastly, F, G, H, I, as much greater than E, as D, C, B, A, are lesser. The Reason is plain, for here E multiplied by the Number proposed is the Sum of the other Numbers found; but E, and the Number proposed, being both Powers of 2, so is the Product or Sum.

PROBLEM II.

To find an even Number of Terms oddly even only, whose Sum is both oddly and evenly even.

Rule 1°. If the given even Number is greater then 2, then take as many Terms as half the Number proposed out of the Series of Powers of 2, beginning at any Power above 4; then take a Number oddly even only below each of these, and another as far above it; and you have the Numbers sought.

Exam. To find 6 such Numbers; they are A, C, D, F, G, I.

A	C	D	F	G	I
6	10	14	18	30	34
	8		16		32
	B		E		H

DEMON. $A+C=2B$. Again, $D+F=2E$, and $G+I=2H$, therefore $A+C+D+F+G+I=2\times\overline{B+E+H}$; but B. E. H. being evenly even only, their Sum $B+E+H$ is both oddly and evenly even (by *Theor.* XXI.) and so also is $2\times\overline{B+E+H}$ (*Cor.* 5. *Theor.* XVIII.) and how many Terms soever you thus find, the Reason is plainly the same; also the Reason why you must begin above 4 is, because there is not a Number oddly even only below 4.

2°. If the given Number is 2, take any Number oddly even only, as A; then a Number greater, as B, which is evenly even only; and a Number oddly even only, C, as far above B as A is below it; then lastly, take D, the Number oddly even only, which is the next above C; and A, D, are the Numbers sought.

Example.

A	B	C	D
6	8	10	14

DEMON. $A+C=2B$, also $C+4=D$, therefore $A+D=A+C+4=2B+4$; suppose next, that $B=2x$, and then $2B+4=4x+4=4\times\overline{x+1}$; but x is an even Number (since B is at least 4) therefore $x+1$ is odd, and therefore $4\times\overline{x+1}$ is both oddly and evenly even.

PROBLEM III.

To find a proposed Number of Terms which are both oddly and evenly even, and whose Sum is evenly even only.

Rule 1°. If the proposed Number is even, take as many Numbers oddly even only, and whose Sum is evenly even only, by *Probl.* II. multiply each of them by some Number evenly even only, and you have the Numbers sought.

Exam. To find 4 such Numbers; they are E, F, G, H.

$$\begin{array}{lllll} & A & B & C & D \\ \text{Oddly even only} & 10+ & 14+ & 18+ & 22=64,\ \text{the 6th Power of 2.} \\ \text{Multiplier} & & 4 & & \\ \hline & 40+ & 56+ & 72+ & 88=256,\ \text{the 8th Power of 2.} \\ & E & F & G & H \end{array}$$

DEMON. Numbers oddly even only, as A, B, C, D, being multiplied by some Power of 2, produce Numbers both oddly and evenly even; but $\overline{A+B+C+D}=$ some Power of 2; therefore $4\times\overline{A+B+C+D}$ is also some Power of 2. Also $4\times\overline{A+B+C+D}=E+F+G+H$, which is therefore some Power of 2, or a Number evenly even only.

2°. If the proposed Number is odd, take the next lesser Number which is even, and find as many Terms both oddly and evenly even, and whose Sum is evenly even only (by *Case* I.) to this Sum add the Number evenly even only, which is the *next greater*, and this last Sum is the remaining Term sought.

Exam. To find 3 Numbers; find 12 and 20, Numbers both oddly and evenly even, whose Sum is 32, evenly even only; to this I add 64, the next greater evenly even, the Sum 96 is the remaining Number sought; for $12+20+96=128=$ the 7th Power of 2.

DEMON.

Demon. Let A, B, C, D, &c. be any Numbers both oddly and evenly even, and whose Sum S is evenly even only, then the next greater Number evenly even only is $2S$. Also their Sum $S+2S=3S=4S-S$, to which add the preceding Numbers found, *viz.* $A+B+C+D$ &c. or their Sum which is S, the total Sum is $4S-S+S=4S$, which is evenly even only, because 4 and S are so.

PROBLEM IV.

To find a Number of Terms, all of them both oddly and evenly even, and whose Sum is both oddly and evenly even.

Rule 1°. If the Number of Terms is even, take (by *Probl.* II.) as many Terms (as the Number proposed) which are all oddly even only, and whose Sum is both oddly and evenly even; multiply them by 2, or by any Number evenly even only, and you have the Numbers sought.

Exam. first, to find 4 Numbers.

Oddly even only,	$6+10+14+18=48$
Multiplier	2
	$12+20+28+36=96$

Numbers sought both oddly and evenly even.

Demon. The Products are Numbers both oddly and evenly even (*Theor.* XVIII. *Cor.* 4.) and the Sum of the Numbers multiplied being both oddly and evenly even, its Product by the same Multiplier is both oddly and evenly even (*Cor.* 5. *Theor.* XVIII.) and is also equal to the Sum of the Numbers formerly produced.

2°. If the Number of Terms is odd, take as many oddly even only, their Sum is always oddly even only; multiply them by 2, or some Number evenly even only, you have the Numbers sought.

Example second.

	$A+B+C=S$
Oddly even only	$6+10+14=30$
Multiplier	2
Numbers sought	$12+20+28=60$

All both oddly and evenly even.

Demon. Any odd Number of Terms, A, B, C, &c. all oddly even only, have a Sum S oddly even only (*Theor.* XX.) and these, or their Sum being multiplied by 2, or any Power of it, produce Numbers both oddly and evenly even. (*Cor.* 5. *Theor* XVIII.) Also the Sum of these Products is the Product of the Sum of the former, *viz.* $A+B+C$, &c. by the same Power of 2, which we have already said is both oddly and evenly even.

PROBLEM V.

To find any Number of Terms, all both oddly and evenly even, with any Number of Terms evenly even only, whose Sum all together is evenly even only.

Rule. Find the Number of Terms proposed both oddly and evenly even, and whose Sum is evenly even only (by *Probl.* III.) Take that Sum as the least of the Terms sought evenly even only; and take the rest of them immediately adjacent to that, and greater, in the Order of the Series of Numbers evenly even only.

Exam. To find 6 Numbers, whereof 3 are both oddly and evenly even, and 3 of them evenly even only; and whose Sum is evenly even only.

Both oddly and evenly even,	$12+20+96=128$ $(=2^7)$
Numbers evenly even only,	$128+256+512$
	Sum$=1024=2^{10}$

Demon. By the Rule of Geometrical Progressions, the Sum of a Progression, whose Ratio is 2 (*i. e.* the Sum of any Number of immediately adjacent Powers of 2) is equal to the Difference betwixt double of the greatest Extreme, (which is equal to the next greater

greater Term in the Progression) and the lesser Extreme (for the Sum of any Geometrical Progression is thus expressed $\frac{rl-a}{r-1}$, but r being 2 the Sum is $2l-a$) therefore if that lesser Extreme be added to the Sum, this Sum is equal to the next greater Term in the Progression; for $2l-a+a=2l$. Now let any Number of Terms both oddly and evenly even, and whose Sum is evenly even only, be A+B+C+D+&c. =M; and let as many Terms evenly even only, be M, N, O, P, &c. the Sum of these last is, 2P−M, to which if we add the Sum of the preceding Numbers, which is M, the Total is 2P, the next greater Number evenly even only.

PROBLEM VI.

To find Numbers, as in the last, whose Sum is both oddly and evenly even.

Rule. Find the proposed Number of Terms, both oddly and evenly even, whose Sum is both oddly and evenly even (by *Probl.* IV.) then take as many Terms evenly even only, any where out of the Series of the Powers of 2.

Exam. To find 3 Terms of each Kind.

	A	B	C	S
Both oddly and evenly even	12+	20 +	28=	60
Evenly even only	64+	128+	256=	448
	M	N	O	R
Sum both oddly and evenly even	508=2×254=4×127			

Demon. Let any Number of Terms both oddly and evenly even, and whose Sum is both oddly and evenly even, be A, B, C, &c. and their Sum S; then take the proposed Number of Terms, all Powers of 2, any where out of that Series, and call them M, N, O, &c. and their Sum R; this Sum is both oddly and evenly even (*Theor.* XXI.) and the total Sum is therefore S+R, both Parts of which being both oddly and evenly even, their Sum is so also (by *Theor.* XXII.)

Theorem XXVI.

Betwixt two Numbers, both even, or both odd (whereof one of them may be 1) there is at least one Arithmetical Mean in Integers.

Demon. The Sum of two even, or two odd Numbers, is an even Number, and consequently is measurable by 2, but the half Sum of the Extremes is the Arithmetical Mean; therefore

Exam. Betwixt 4 and 6, there is one Mean, 5; and betwixt 5 and 7 there is one Mean, 6.

Theorem XXVII.

Betwixt an even Number, and an odd (which may be 1) there are at least 2 Arithmetical Means in Integers, or there are none at all; nor can there possibly be any odd Number of Means.

Demon. The Sum of an even and odd Number is odd, therefore they do not admit of one Arithmetical Mean, because the Sum being odd is not measurable by 2, consequently there must be 2 Means at least if there are any. Hence again, there cannot be an odd Number of Means; for then there would be one odd Mean, contrary to what's last shewn.

Theorem XXVIII.

If a Geometrical Progression is in its lowest Terms, they are either all odd Numbers, or all even, except one of the Extremes, which must be odd.

Demon.

Demon. Let $A:B$ be the loweſt Term of the Ratio of any Geometrical Progreſſion; then will the Series be thus repreſented, $A^n : BA^{n-1} : B^2A^{n-2} : B^3A^{n-3}$: &c. $AB^{n-1} : B^n$ (*Probl.* VI. *Schol.* 1. *Ch.* I.) Now A and B are either both odd, or one odd and the other even; for if they are both even they are not in the loweſt Terms: But all the Powers of odd Numbers are odd, and of even Numbers are even; and the Product of an odd Number by an odd, is odd, and of an even by an odd, is even: Whence the *Theorem* is manifeſt.

THEOREM XXIX.

If an even Number is a Square, it has an even Half, or is meaſurable by 4; but if an odd Number is a Square, then being divided by 4, it leaves a Remainder of 1; or 1 taken from an odd Square, leaves a Multiple of 4.

Demon. 1°. Any even Root may be expreſſed $2n$; and its Square will be $4n^2$, *viz.* a Multiple of 4; which ſhews the firſt Part. Again,

2°. Any odd Root may be expreſſed $2n+1$, and its Square will be $4n^2+4n+1$, *viz.* $4\times\overline{n^2+n}+1$; which is plainly a Multiple of 4, and 1 remaining over.

COROLLARIES.

1*ſt.* The Sum of any Number of even Squares is meaſurable by 4 (or has an even Half.)

2*d.* The Sum of 2 or 3 odd Squares, divided by 4, leaves a Remainder of 2 or 3. *Univerſally*, if the Number of odd Squares added, is a Multiple of 4 (as 4 . 8 . 12 . 16 . 20 &c.) the Sum is meaſurable by 4; otherwiſe there will always be a Remainder; particularly if that Number is the Sum of a Multiple of 4, and 1, or 2, or 3; the Remainder will be accordingly, 1, or 2, or 3.

3*d.* The Sum of an even and odd Square, divided by 4, leaves a Remainder of 1; and *univerſally*, if any Number of even Squares is added to any Number of odd Squares, the Remainder will be the ſame that would happen with the Sum of the odd Squares; becauſe the Sum of the even Squares leaves no Remainder.

4*th.* The Sum of any two integral Squares, being divided by 4, cannot leave a Remainder of 3; for if they are both even, the Remainder is *o*; ſince each of theſe Squares is meaſurable by 4, by this *Theorem*; and conſequently their Sum is ſo. If the one is even, and the other odd, the Sum will leave a Remainder of 1 (*Coroll* 3.) *Laſtly*, If both are odd, the Remainder will be 2 (*Coroll.* 2.)

THEOREM XXX.

The Terms of an Arithmetical Progreſſion are either all even or all odd; or they are alternately even and odd; *i. e.* the 1ſt, 3d, 5th, &c. Terms are all even or odd; and the 2d, 4th, 6th, &c. all odd or even. Alſo the Sum of the Whole is odd or even, according as the Number of odd Terms is odd or even; but if all the Terms are even the Sum is even.

Demon. This depends all upon the leſſer Extreme, and the Difference compared with *Theorem* I. Thus,

1°. If the leſſer Extreme and Difference are both even, ſo muſt the whole Series be; becauſe even Numbers are ſtill added to even. So if the leſſer Extreme is 2, and the Difference 4, the Series is 2 . 6 . 10 . 14, &c.

2°. If the 1ſt Term is odd, and the Difference even, all the Terms are odd; becauſe they are each the Sum of an even and odd Number. So the 1ſt Term being 3, and the Difference 4, the Series is 3 . 7 . 11 . 15, &c.

3°. If the 1ſt Term is even and the Difference odd, or if both are odd, the Terms are alternately odd and even; becauſe an odd and even makes the Sum odd; and two odds make an even.

4°. That

4°. That the Sum of the Series will be odd or even, according as the Number of odd Terms is odd or even, is also evident from the same Principles; for the Sum of every two odd Numbers is even: So that if the Number of odd Terms is even, the Sum of them, and consequently of the whole Series, is even; but if it's odd, the Sum is odd; which added to the even Sum of the even Terms, makes the total Sum odd. If all are even, the Sum is manifesty even.

THEOREM XXXI.

Take the odd Series, 1 . 3 . 5 . 7 . 9 . &c. The Sum of any even Number of Terms of this Series, taken in the continued Order of the Series, and beginning at any Term, is a Number both oddly and evenly even (*i. e.* it has an even Half; or is measurable by 4.) For *Exam.* $5+7=12=3\times4$, and $5+7+9+11=32=8\times4$.

DEMON. 1°. The Sum of any two adjacent Terms is measurable by 4; for it is equal to the Double of that Term of the natural Series which lies betwixt them, and is the Arithmetical Mean; but that Mean is an even Number; and double of an even Number is evidently measurable by 4; or is an even Number with an even Half.

2°. Since this is true of any two adjacent Terms, it must be true of any even Number of adjacent Terms; because these being distributed into 2's and 2's, the Sum of each 2 is measurable by 4; consequently the Sum of the Whole is measurable by 4.

THEOREM XXXII.

If out of the odd Series, 1 . 3 . 5 . 7 . 9 . &c. be taken in the continued Order of the Series, any odd Number of Terms, beginning at any Term, the Sum of them is an odd Number; whose Place, in the same Series, has this constant and regular Connection with the Number of Terms, and the Place of the lesser Extreme of the Terms added, *viz.* that, if you take the Product of that Number of Terms, by the Place of the lesser Extreme; then again, Take the Half of the Square of the Number of Terms less 1; add this Half Square to the former Product; the Sum is the Place of the Sum of the Terms added.

Thus, if there are 3 Terms added, and the Place of the least be n, the Place of the Sum is $3n+2$. If there are 5 Terms added, it is $5n+8$; and so on, as in this Table.

$3n+2$ $(=2\times1)$
$5n+8$ $(=2\times4)$
$7n+18$ $(=2\times9)$
$9n+32$ $(=2\times16)$
&c.

DEMON. 1°. That the Sum is an odd Number is already proved in *Theorem* I.

2°. That the Place of the Sum is according to the *Theorem*, is deduced from the Rules concerning Arithmetical Progressions; Thus,

Call the Place of the lesser Extreme of the Series added, n; then that Term it self is $2n-1$ [from the Nature of the Series; for it is $1+\overline{n-1}\times2=1+2n-2=2n-1$.] Let the Number of Terms added be a; the greatest Extreme added must be $2n-1+2\times\overline{a-1}=2n+2a-3$ [for $2n-1$ is the lesser Extreme, 2 the common Difference, and a the Number of Terms] then the Sum of the Extremes is $\overline{2n-1+2n+2a-3}=4n+2a-4$; and the total Sum is $\overline{4n+2a-4}\times\frac{a}{2}=2an+aa-2a$. Now this being a Term of the odd Series, 1 . 3 . 5 . &c. suppose its Place to be N; then that Term of the odd Series, whose Place is N, is it self $2N-1$ (as above shewn for the Place n;) so that $2N-1=2an+aa-2a$: Add 1 to both, and then divide by 2, and it is $N=\frac{2an+aa-2a+1}{2}=an+\frac{aa-2a+1}{2}$. But $\overline{a-1}^2=a^2-2a+1$, and the Half of this is $\frac{aa-2a+1}{2}$; whence the Rule is evidently demonstrated.

SCHOL.

SCHOL. The half Squares of the Number of Terms less 1, are the Products of the Series of Square Numbers, 1 . 4 . 9 &c. multiplied by 2, as I have marked them in the Table; and that it will go on so for ever, will be plain, thus; Let a be any odd Number, the next greater odd Number is $a+1$: Take 1 from each of them, the Remainders are $a-1$, $a+1$; whose Squares are a^2-2a+1, a^2+2a+1; whose Halfs are $\frac{aa-2a+1}{2}$, $\frac{aa+2a+1}{2}$: But if the first of these is the Double of any square Number, the other must be double of the next greater Square; for suppose $\frac{aa-2a+1}{2}=2\times b^2$, then is $aa-2a+1=4\times b^2=\overline{2b}^2$; hence $a-1=2b$. Add 2 to each, and it is $a+1=2b+2=2\times\overline{b+1}$; therefore $\overline{a+1}^2$, or $a^2+2a+1=4\times\overline{b+1}^2$, and $\frac{aa+2a+1}{2}=2\times\overline{b+1}^2$.

§. V. *Of Numbers*, Perfect, Abundant, *and* Deficient.

THEOREM XXXIV.

IF the Geometrical Progression, 1 : 2 : 4 : 8, &c. is carried on till the Sum be a prime Number; and that Sum be multiplied by the last Term of the Series, the Product shall be a *perfect* Number; thus, $1+2=3$, a Prime, and $3\times 2=6$, a perfect Number. Again, $1+2+4=7$ and $7\times 4=28$, a perfect Number; for its *aliquot* Parts are $1+2+4+7+14=28$.

DEMON. Let $1+2+4+8+$ *&c.* $+2^n=S$, a prime Number; then is $S\times 2^n$ a perfect Number: For,

1°. If from S we raise a Series in the Ratio 1 : 2, having as many Terms as the preceding; the last Term of it will be $S\times 2^n$, as is evident from the Method of raising the Series.

1, 2, 4, 8, &c. 2^n	
S, 2S, 4S, 8S, &c. $2^n\times S$	

2°. It's evident from the Composition of these Numbers, that all the Terms of both these Series, from 1 and S, are *aliquot* Parts of $S\times 2^n$; for the first Series after 1, are all Powers of the same Root, 2; which therefore measure 2^n, and consequently $2^n\times S$; and the second Series being only the Multiples of the first by S, therefore each of them also measures $2^n\times S$.

3°. By *Probl.* IV. *B.* IV. *Ch.* III. The Sum of all the Terms of a Geometrical Progression, excluding the greatest Extreme, is the Quote of the Difference of the Extremes, divided by the Ratio less 1; but the Ratio here being 2, therefore $S+2S+4S+8S+\&c.=2^n\times S-S$, and $1+2+4+8\ \&c.+2^n=S$; also $2^n\times S-S+S=2^n\times S$; therefore $1+2+4+8\ \&c.+2^n+S+2S+4S+8S\ \&c.=2^n\times S$.

4°. It being proved that $2^n\times S$ is the Sum of all the other Numbers in these two Series, and that each of these are *aliquot* Parts of it; it remains to shew that no other Number can be an *aliquot* Part of it. Thus, Every other Number must necessarily have in its Composition some other Prime than 2 or S, or some higher Power of one or both of these than is in $2^n\times S$; but by *Theor.* X. no such Number can measure $2^n\times S$, or can be an *aliquot* Part of it.

SCHOL. In this *Theorem* there is a certain Way of finding as many perfect Numbers, as the Number of Cases in which the Sum of the Series, 1 . 2 . 4 . 8 &c. can be a prime Number; in which *observe*, that there is no more to do, but from every Term of the Series, as it goes on, to take 1; the Difference is the Sum of all the preceding Terms; and if it's a Prime, then being multiplied into the preceding Term, it gives a perfect Number. That there are some perfect Numbers found this Way, is certain;

for such a Number is $6 = \overline{1+2}\,(=3) \times 2$; also $28 = \overline{1+2+4}\,(=7) \times 4$; and $496 = \overline{1+2+4+8+16}\,(=31) \times 16$; and $8128 = \overline{1+2+4+8+16+32+64}\,(=127) \times 64$. And $33550336 = \overline{1+2+4+8+16+32+64+128+256+512+1024+2048+4096}\,(=8191) \times 4096$. Besides these there are but a few more found. Dr. *Harris* says there are but ten perfect Numbers betwixt 1 and 1,000,000,000,000, but does not express them. And *Tacquet* observes, that the Reason why more are not found, is, That in the Progression, 1 . 2 . 4 . &c. the Intervals of the Numbers, which lessened by an Unit, become Primes, are very great; and when the Numbers are great, the finding whether they are Primes or not is a vast Labour. There is one Thing more I would observe here, that it has not yet been demonstrated, as far as I know, that there can be no perfect Numbers but what may be found by this *Theorem*, *i. e.* that every perfect Number is the Product of two Numbers, whereof one is a Prime, and the Sum of a certain Number of Terms of the Series, 1 . 2 . 4 . &c. the other, the last of these Terms. Again, it wants also to be demonstrated, that the Number of perfect Numbers is infinite.

THEOREM XXXV.

Every Prime Number is a deficient Number.

DEMON. A Prime has no *aliquot* Part but 1; which is less than any prime Number.

THEOREM XXXVI.

Every Power of 2 is a deficient Number.

DEMON. No Number can measure or be an *aliquot* Part of any Power of 2, but 1, or the inferior Powers of 2 (*Coroll.* 6. *Theor.* X.) But any Power of 2, is more by 1 than the Sum of all the inferior Powers and 1 (by the Rule for summing the Geometrical Progression, 1 . 2 . 4 . &c.) and therefore is a deficient Number.

THEOREM XXXVII.

Every Number is *abundant*, which is measured by a perfect or an abundant Number: Or thus, A perfect or abundant Number can measure no Number but an Abundant.

p	n
a	r
b	s
&c.	&c.
1	u

DEMON. Let p, a Number perfect or abundant, measure n; and let a, b, &c. 1, be the *aliquot* Parts of p; and take r, s, &c. u, in the same Ratios to n, as a, b, &c. 1 are to p; they will all be Integers; for since $p : a :: n : r$, then $p : n :: a : r$; but p measures n, therefore a measures r, which consequently must be Integer; and so of the rest. Now because $p : n :: a : r :: b : s :$ &c. $:: 1 : u$, therefore $p : n :: a+b$ &c. $+1 : r+s$ &c. $+u$; but $a+b$ &c. $+1$, is either equal to, or greater than p, as this is a perfect or abundant Number: And hence, $r+s$ &c. $+u$ (which are all *aliquot* Parts of n, because $a+b$ &c. are so of p) is equal to, or greater than n; if greater, the *Theorem* is demonstrated; if equal, then 1, which is an *aliquot* Part of n, different from any of these, r, s, &c. u, being added to them, makes the Sum greater than n; which is therefore abundant.

COROLLARIES.

1*st*. A perfect or deficient Number can be measured only by a Deficient; because what is measured by a Perfect or Abundant is Abundant.

2*d*. An abundant Number may be measured by any Number; for the Multiple of any abundant or perfect Number is an abundant Number; and what is measured by a perfect Number is measured by all the *aliquot* Parts of it, which are all deficient Numbers (by the 1st *Coroll.*) Hence again:

3*d*. A

3*d*. A deficient Number measures any kind of Number.

Schol. As in *Theorem* XXXIV. we have a Rule for finding perfect Numbers, so from this *Theorem* we have a Rule for finding abundant Numbers; for such are all the Multiples of any perfect Number. And from *Coroll.* 1. we have a Rule for finding deficient Numbers; all the *aliquot* Parts (except 1) of any perfect Number being such.

Exam. 1. 6 is perfect, and 18 is its Multiple, whose *aliquot* Parts are $1+2+3+6+9=21$.

Exam. 2. 28 is Perfect, and 14 its Half, whose *aliquot* Parts are $1+2+7=10$.

Theorem XXXVIII.

If any Number, A, multiplying a Prime, p, produces a perfect Number, N; the same A multiplying another Number M, which is less than p, and which does not measure A, produces an abundant Number, O.

Exam. $4\times7=28$, a perfect Number, and $4\times6=24$, an abundant Number, whose *aliquot* Parts are $1+2+3+4+6+8+12=36$.

	N	O	
A	p	M	
a	r	m	x
b	s	n	
&c.	&c.	&c.	
1			

Demon. Let a, b, &c. 1, be all the *aliquot* Parts of A; then because $N=A\times p$, a, b, &c. will measure N; and because p is a Prime, N has no other Measure except A, a, b, &c, 1. and the Products of these by p (as you see in *Probl.* IV. §. I.) But it's also plain, that N being divided by a, b, &c. the Quotes are also *aliquot* Parts of N, and therefore must be the same Numbers as the Products of p by a, b, &c. though not answering in the same Order, *i. e.* if $\frac{N}{a}=r$, this is not the same as ap; but as they must necessarily be the same Numbers, however the Correspondence be, let us suppose $\frac{N}{a}=r$, $\frac{N}{b}=s$, &c. Then again, since $O=AM$, therefore a, b, &c. which measure A, do also measure O; let $\frac{O}{a}=m$, $\frac{O}{b}=n$, &c. wherefore $N:O::p:M::r:m::s:n$, &c. and compoundly, $N:O::\overline{p+r+s}$, &c. $:\overline{M+m+n}$, &c. But $\overline{A+a+b}$ &c. $\overline{+1+p+r+s}$ &c. $=N$; so that $\overline{A+a+b}$ &c. $\overline{+1}$ is the Remainder, after $\overline{p+r+s}$ &c. is taken out of N; let x be the Remainder after $\overline{M+m+n}$ &c. is taken out of O (which must be greater than that Sum, since N is greater than $\overline{p+r+s}$ &c.) then is $N:O::A+a+b$, &c. $+1:x$; but M being less than p, O is less than N; and consequently x is less than $\overline{A+a+b}$ &c. $\overline{+1}$; also $\overline{M+m+n}$ &c. $+x=O$; therefore $\overline{M+m+n}$, &c. $\overline{+A+a+b}$, &c. $\overline{+1}$ (each of which measures O) is greater than O. And since, lastly, M does not measure A, therefore M, m, n, &c. A, a, b, &c. 1, are all different *aliquot* Parts of O, which is therefore Abundant.

Theorem XXXIX.

If a Number, A, multiplied into another, B, produces either a perfect or abundant Number; then if A is multiplied into any Multiple of B, the Product is Abundant.

Exam. $2\times3=6$, a perfect Number, $2\times5=10$, and $3\times10=30$, an abundant Number, whose *aliquot* Parts are $1+2+3+5+6+10+15=42$.

Demon. Let M be a Multiple of B; then is $B:M::AB:AM$. And because B measures M, so does AB measure AM; but AB is, by Supposition, *Perfect* or *Abundant*; therefore (by *Theor.* XXXVII.) AM is *Abundant*.

CHAP. II.

Of Figurate Numbers.

§ 1. *Definitions.*

I. NUMBERS are called *Figurate* from Geometrical Figures, which they are capable of representing in a certain Manner, by a particular Disposition of their Units (as shall be presently explained;) which is a Part of the antient *Pythagorean* Speculations about Numbers and Geometrical Figures; from the Comparison of which they found such Likenesses and Correspondencies, whence they pretended to discover many Mysteries and Secrets of Nature. Our Business here is to consider these Numbers as a Subject purely Arithmetical, and upon the Principles of Numbers only to explain their Connections and Properties; yet it being necessary to have Names for Things, and simple Names being more convenient than long Descriptions; and the Geometrical Names (described below) being still in use, we shall retain them, and explain the Reason and Meaning of them, for their sake who have Acquaintance enough with Geometry to understand it, or Imagination to conceive it by the following Explications; for others, they must take them as mere Names, by which these Numbers are designed and distinguished.

II. Take any Arithmetical Progression, beginning with 1, and whose common Difference is any integral Number; then take the Sums of these Series continually from the beginning; and again, the Sums of these Sums, and so on for ever. These several Series of Sums are called in general Figurate Numbers, but more particularly, the first Sums are called plain Figurates, and also Polygons; the second Sums are called solid Figurates, and also Pyramids; the third Sums are called second Pyramidals, and so on. But again,

III. Polygons are distinguished thus, If the common Difference in the Series $\div l$, whence they proceed, or whose Sums they are, is 1. as 1 . 2 . 3 . &c. the Sums 1 . 3 . 6 . &c. are called Triangles. If the Difference is 2, as 1 . 3 . 5 . &c. the Sums 1 . 4 . 9 . &c. are called *Quadrangles*, and particularly Squares. If the Difference is 3, as 1 . 4 . 7 . &c. the Sums 1 . 5 . 12 . &c. are called *Quinquangles* or *Pentagons*, and so on; the Name of the Polygon expressing a Figure of a Number of Angles, which is 2 more than the common Difference of the Series $\div l$. In the same Manner,

IV. Pyramids, and all the following Sums, are distinguished by the Polygon whence they proceed; and thus we have Triangular Pyramids, Square Pyramids, &c. also Triangular and Square, second Pyramidals, third Pyramidals, and so on.

V. Since the Pyramidals do all proceed from Polygons, they may also be called Polygonal Numbers; and the whole Order of Sums be more conveniently distinguished, by calling them Polygonals of the first, second, &c. Order: Thus, the first Sums or Polygons, are Polygonals of the first Order; the second Sums, or Pyramids, are Polygonals of the second Order. And again, for the several Orders proceeding from different Series $\div l$, they are to be distinguished by the Name of the Polygon, which is particularly applied to the first Order, and so on. Thus all the Order of Sums proceeding from the Series 1 . 2 . 3 . &c. are Triangulars of the first or second, &c. Order. These from the Series 1 . 3 . 5 . &c. are Polygonals of the square Kind, and so on. *Observe again*, That instead of these Names Triangular, &c. it will be sometimes more convenient

venient to diſtinguiſh them by firſt Species, ſecond Species, *&c.* and then the Arithmetical Denominations of firſt, ſecond, *&c.* being the ſame Numbers as the common Differences of the Series $\div l$, theſe are clearly marked by this Denomination; and thus as all the different Series of Sums come under the general Name of Polygonal Numbers, ſo theſe from different Series $\div l$ are diſtinguiſhed by different *Species*, and the different Series of Sums proceeding from the ſame Series $\div l$ are diſtinguiſhed by different Orders. But in the laſt Place obſerve, that we ſhall ſometimes uſe the ſimple Name Polygon; or alſo particularly, Triangle, Square, *&c.* when we ſpeak of the firſt Sums, or firſt Order of *Polygonals*; alſo the ſimple Name *Pyramid* for the ſecond Sums, or Sums of Polygons.

I ſhall now repreſent all theſe Series in diſtinct Tables, according to their Species and Orders; and then explain the Reaſon of the particular Names.

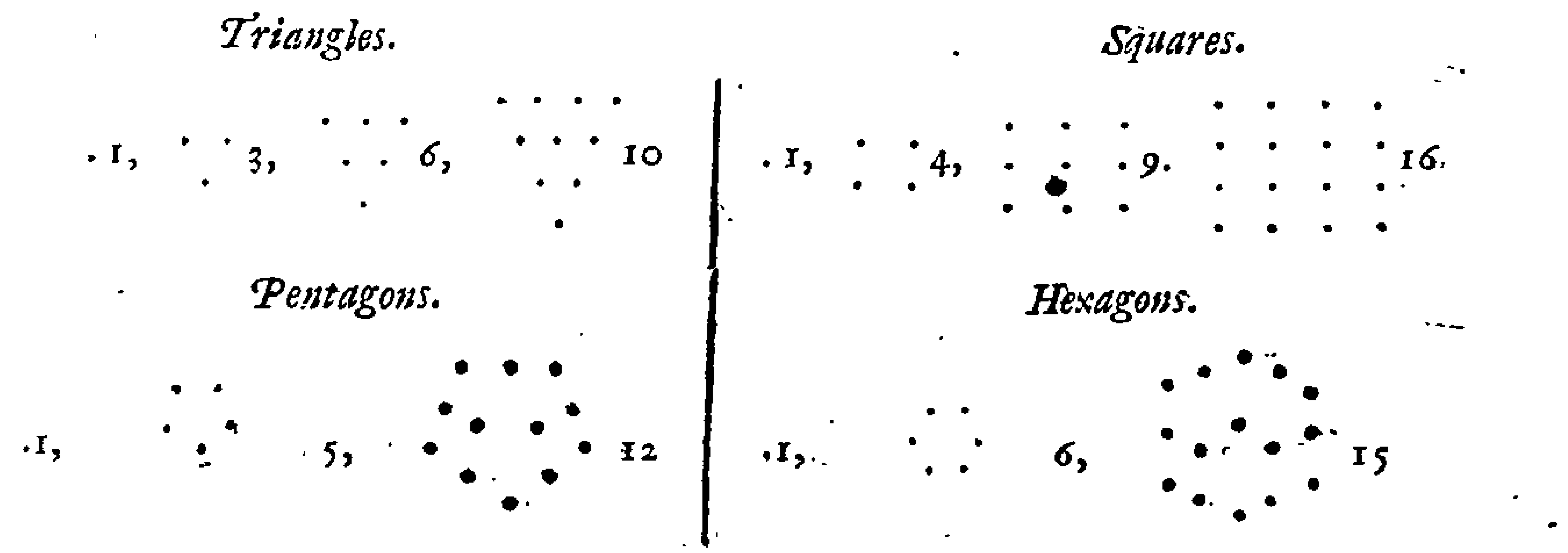
Table of Polygonal *Numbers.*

Series $\div l$		*Polygons*, or *Polygonals*, *1ſt Order.*			*Pyramids*, *Polyg. 2d Order.*		*2d Pyramidals* *Polyg. 3d Order.*	
1. 2. 3. 4		Triangles,	1. 3. 6. 10		1. 4. 10. 20		1. 5. 15. 35	
1. 3. 5. 7		Squares,	1. 4. 9. 16		1. 5. 14. 30		1. 6. 20. 50	
1. 4. 7. 10	*&c.* 1ſt Sums.	Pentagons,	1. 5. 12. 22	*&c.* 2d Sums.	1. 6. 18. 40	*&c.* 3d Sums.	1. 7. 25. 65	*&c.*
1. 5. 9. 13		Hexagons,	1. 6. 15. 28		1. 7. 22. 50		1. 8. 30. 80	
1. 6. 11. 16		Heptagons,	1. 7. 18. 34		1. 8. 26. 60		1. 9. 35. 95	
1. 7. 13. 19		Octogons,	1. 8. 21. 40		1. 9. 30. 70		1. 10. 40. 110	
&c.			*&c.*		*&c.*		*&c.*	

The Reaſon of the Names.

A Number is called a *Polygon*, from the Repreſentation of a plain Figure having many Angles, and ſuch too as is Regular, or has equal Angles, and equal Sides. Thus *Triangles* repreſent Equiangular *Triangles*, Squares, equal angled *Quadrangles*, and ſo on. Which Repreſentation you ſee in the following Schemes; wherein 1 is of all Species, becauſe every Thing is an Unit of its Kind.

Polygons.

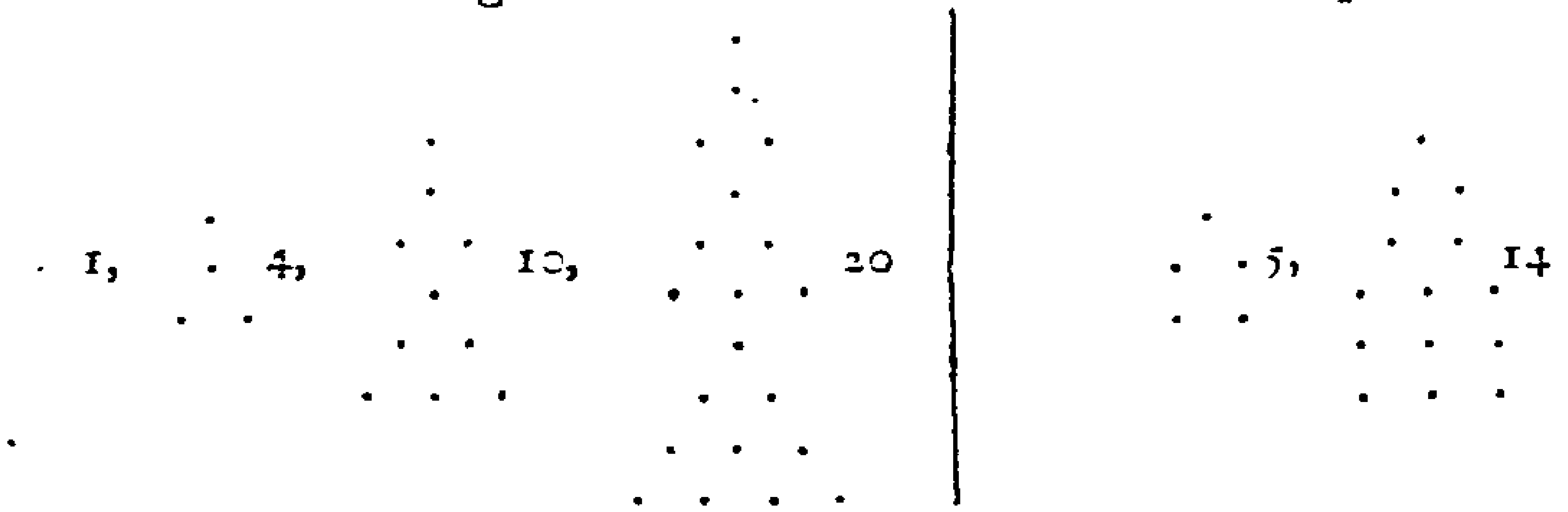


Such is the Diſpoſition of the Units of theſe Numbers, from whence they are called *Triangles*, &c. and ſo will the Repreſentation go on as it is here begun, both as to the

the Continuation of these here represented, and all the other Species. But as I meddle no further with these Speculations, so I shall say no more for the Demonstration of it; only this one Thing I must here observe, That the Sums of the Series 1 . 3 . 5 &c. *viz.* 1 . 4 . 9 &c. are not only square Numbers, but they are the Series of Squares of the natural Progression, 1 . 2 . 3 &c. So far as the Series is here carried you see the Truth of the Observation; and that it must continue so for ever may be easily perceived from the Consideration of the Numbers, and the Manner of disposing their Units. But I shall not leave the Demonstration of it merely upon this; in another Place I shall propose and demonstrate it distinctly by it self; and till then, consider these Numbers only as the Sums of the Series 1 . 3 . 5 &c.

Pyramids.

Triangular. *Square.*



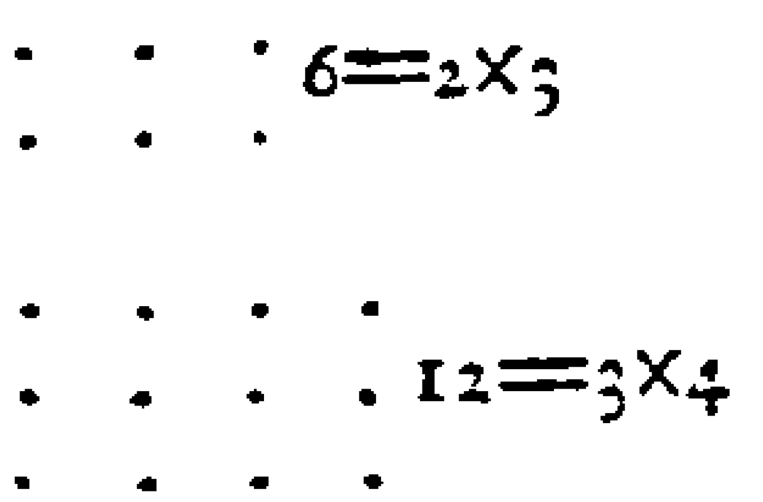
By conceiving the Planes of each of the Polygons which compose a Pyramid to be placed parallel over one another, and disposed, with respect to the Situation of their Angles and Distances, so that the respective Angles of each Polygon be in a right Line with one another, and with the vertical Point or Unit; this does in a Manner represent a *Pyramid*, and hence the Name.

The other Orders of Pyramidals have no such Representation, and are mere Combinations of the preceding, called Pyramidals only for a Distinction from the Pyramids whence they proceed.

VI. The Place of any Term in any Series of *Polygonals*, which is the Number of Places from the beginning to that Term, is called the *Root* or *Side* of that Polygonal; because in the Polygon, represented it is the Number of Points or Units that makes the Side of the Figure; so 10 is the 4th Term of the Triangles, and 20 the 4th Term of the Triangular Pyramids; wherefore 4 is called their Root or Side; or we may as well call it the Place of any Term.

VII. Polygonals that stand in the same Places of their respective Series, are called *Collaterals* (*i. e.* having the same Side.)

VIII. The Product of any two Numbers is called also a plain Figurate Number; and is particularly a Quadrangle, because it can represent such a Figure; and the two Factors are called the Sides of the Figure, as in the annex'd Examples. And observe, that though Squares are Quadrangles, yet because every Quadrangle is not a Square, therefore they may be distinguished by applying the general Name Quadrangle to all the Species excepting Squares. But the Difference will be better marked by distinguishing them into *Oblongs* and *Squares*. Yet again, observe, that the Name Oblong is more particularly applied to that kind wherein



wherein the Sides differ by 1, which are the only Oblongs we consider here, because of their Connection with the Figurates above described; the whole Series of which Oblongs is made out by taking the natural Series 1 . 2 . 3, *&c.* and multiplying each Term into the next, as here,

	1	2	3	4	5	6	*&c.*
Oblongs	2	6	12	20	30		*&c.*

IX. The Product of any 3 Numbers is called also a solid Figurate Number, and particularly a Prism; and yet more particularly it's a Quadrangular Prism; for the Product of two Numbers is a Quadrangle, and the Product of this by the remaining Factor makes a Prism; because by taking any Quadrangle (or other plain Figure) a certain Number of Times, and conceiving them all placed parallel to one another at equal Distances, and so situated, that their respective Angles are in a right Line, they do in a manner represent what in Geometry is called a Prism. But again, in this Doctrine of Figurates, if we take the Product of any of the above described Polygons multiplied by its Side, that is called a Prism (though some of them are not composed of 3 Factors; and such as are so, yet are not considered in that Manner.)

These Prisms are also distinguished by the Polygon whence they proceed. Again, taking the Sums of these Series of Prisms, and the Sums of those Sums, and so on, we have new Series, which may be called in general *Prismatick Numbers*, to be distinguished the same Way as *Polygonals*, by different Orders and Species, as in the following Tables.

Again, *Prisms* being multiplied by their Sides, produce a new Kind of *Prismaticks*; and these again multiplied by their Sides, produce another Kind, and so on; all which we may distinguish by the Names of different Degrees, calling the Products of Polygons by their Sides, *Prismaticks* of the first Degree; the Products of these again by their Sides, *Prismaticks* of the second Degree, and so on. Observe also, That these several Degrees of Prismaticks are the Products of their Polygons by such a Power of their Sides as expresses that Degree; for a being any Polygon, and n its Side, the Prismaticks of the several Degrees proceeding from this Polygon are $a\times n$. $a\times n\times n=a\times nn$. $a\times nn\times n=a\times nnn$. &c. Again, the Sums of Prismaticks of any Degree make also different Orders of Prismaticks of that Degree. *Lastly*, By the simple Name of Prisms always understand the first Degree, or Product of Polygons by their Sides.

Polygons.		*Prisms*, or *Prismaticks* of the 1*st* *Degree* and 1*st* *Order.*		*Prismaticks*, 1*st* *Degree*, 2*d* *Order.*
Triang.	1.3. 6.10	*Triang.*	1. 6. 18. 40	1. 7. 25. 65
Square	1.4. 9.16	*Cubes*,	1. 8. 27. 64	1. 9. 36.100
Pentag.	1.5.12.22	*Pent.*	1.10. 36. 88	1.11. 47.135
Hexa.	1.6.15.28	*Hex.*	1.12. 45.112	1.13. 92.204
	&c.		*&c.*	*&c.*

	2d Degree, 1st Order,	*2d Degree, 2d Order.*
Triang.	1.12. 54.160	1.13. 67.227
4th Powers	1.16. 81.256	1.17. 98.354
Pent.	1.20.108.352	1.21.129.481
Hex.	1.24.135.448	1.25.160.608
	&c.	*&c.*

SCHOL.

SCHOL. We have observed already, That the Polygons of the second Species are the Series of Squares of the Progression 1 . 2 . 3 &c. (which shall be demonstrated afterwards.) And now from this and the Construction of Prismaticks, it follows, that these of the second Species, and 1st Order of all the Degrees successively, are the several Series of the superior Powers of the same Progression 1 . 2 . 3, &c. Thus, those of the first Degree are Cubes or third Powers; and universally, those of the n Degree are $n+2$ Powers. Therefore the whole Doctrine of Powers and Roots may be considered as a Part of this of Figurate Numbers; but as the calling them Figurates proceeds from a Consideration which is not properly Arithmetical; so the Order and Connection of Things in Arithmetick required that this Part concerning Powers and Roots, which is the most useful and necessary, should be particularly handled in another Place, as it is in *Book* III. and several Properties of these and other Composite Numbers (which are all Figurates) you have in *Chap.* I. of this Book.

We proceed now to explain the Properties of such Figurates as have not been yet handled, and some remaining Properties of those that have been in part considered already.

§ 2. *Of* Polygonal *Numbers.*

THEOREM I.

EVery Number is a Polygonal of every Species, and also of every Order whose Denominations are less than it by 2, or by any greater Number.

Exam. 5 is a Polygonal of the Third, and of all the preceding Species and Orders.

DEMON. The first Term in every Species and Order being 1; the second Terms in the several Orders of the same Species, and of the same Order in all the different Species, are, by the Construction, in Arithmetical Progression, with the common Difference 1. Again, the second Term of the first Species in every Order is more by 2 than the Denomination of that Order, and is the least Number, except 1, of all the other Polygonals of that Order; comparing these Things, the Truth proposed is manifest.

LEMMA.

Let any Number of different Series, $a.b.c.d.$ &c. $e.f.g.h.$ &c. as in the Margin, be such, that each collateral Column, as $d, h, m. q.$ &c. is an Arithmetical Progression. Also, let A B . C . D &c. be the Sums of the former, thus, $a=$A. $a+b=$B. and so on. Then are the Collaterals of this last Table also $\div l$, and their common Difference is the Sum of the Differences of all the collateral Columns of the first Table backwards, from that which is in the same Place with any given Column of the second. Thus, let $e-a=x$. $f-b=y$. $g-c=z$, and $h-d=v$, whose Sum call S; then is D . H . M Q in the common Difference S, or H$-$D$=$S ($=x+y+z+v$.)

a	b	c	d	A	B	C	D
e	f	g	h	E	F	G	H
i	k	l	m	I	K	L	M
n	o	p	q	N	O	P	Q

DEMON. $\overline{e+f+g+h}=\overline{a+b+c+d}+$S; for S$=\overline{e-a}+\overline{f-b}+\overline{g-c}+\overline{h-d}$; also $i+k+l+m=e+f+g+h+$S, and so of the other Series; then because D$=a+b+c+d$, H$=e+f+g+h$. M$=i+k+l+m$. Q$=n+o+p+q$, therefore 'tis evident that H$=$D$+$S. M$=$H$+$S. Q$=$M$+$S, and so on: The Reason is the same, how large soever the Tables are, and which soever Column we chuse.

THEOREM II.

The Collaterals of the ſeveral Series $\div l$ (whoſe Sums make Polygons) are alſo $\div l$, whoſe common Difference is the preceding Term of the firſt Line; ſo 4 . 7 . 10 . 13 have the common Difference 3.

Again, the Collaterals of the ſeveral Species of any Order of Polygonals, are $\div l$, whoſe Difference is the preceding Term of the firſt Line or Triangular Species; ſo in the firſt Order, 10 . 16 . 22 . 28. differ by 6.

DEMON. 1°. For the Collaterals of the Series $\div l$, the firſt and leaſt Term of each Series is 1; and calling the Difference in any Series d, the n^{th} Term is $1+\overline{n-1}\times d$ (by the Rules of Progreſſions $\div l$.) But in the ſeveral Series the Differences are gradually 1 . 2 . 3 *&c.* wherefore in the Expreſſion $1+\overline{n-1}\times d$, d being ſucceſſively 1 . 2 . 3 *&c.* it follows that the Collaterals expreſſed univerſally $1+\overline{n-1}\times d$, are theſe $1+\overline{n-1} : 1+\overline{n-1}\times 2 : 1+\overline{n-1}\times 3$ *&c.* which is a Series in the Difference $n-1$, which is the Term of the firſt Series preceding n, the Place of the Collaterals.

2°. For the Collaterals of the ſeveral Orders of Polygonals, the Theorem follows plainly from the preceding Lemma: For the Polygonals of the 1ſt Order proceed from the Series $\div l$, whoſe Collaterals are $\div l$, therefore, by the Lemma, theſe laſt are alſo $\div l$, and their Differences are the Sums of the Differences in the correſponding, and all the preceding Columns of the other: Alſo theſe other Differences being the preceding Term of the firſt Line, their Sum is the preceding Term of the firſt or Triangular Species of the firſt Order of Polygonals, or of the ſimple Polygons. For the ſame Reaſon, the Thing propoſed is true in the ſecond, and all the following Orders of Polygonals.

COROL. A Polygonal Number of any Order and Species is equal to the Sum of the Collateral Polygonal of any preceding Species of the ſame Order, and the Product of the Diſtance of theſe two Species (*i. e.* the Number of the Species, leſs 1, from the one Species to the others) multiplied by the preceding Polygonal of the firſt Species. This is manifeſt, becauſe it is nothing but the Rule for expreſſing the greateſt Term of a Series $\div l$, by means of the leſſer Term, the Number of Terms, and the common Difference. Thus, for Example, in the firſt Order, 28 is in the 4th Place of the 4th Species, and 28 = the Sum of 16 (the 4th Term of the 2d Species) + 12 (the Product of 6, the preceding Triangular, and 2, the Diſtance of the 4th and 2d Species.)

General SCHOLIUM.

In order to find the Sum of any Series of Polygonals, or any Term of any Series of Polygonals, it is plain that we want only a Rule for finding any Sum, or Term of any Series of the 1ſt or Triangular Species; becauſe thereby we can find any Term of the Collaterals of the Order given, by Coroll. preceding. *Again obſerve*, That any Polygonal of the firſt Order being the Sum of an Arithmetical Progreſſion, we know how to find any of theſe by the Rules of Progreſſions, and what we want is a Rule for the other Orders; but there is one general Rule which comprehends them all; in order to the Inveſtigation of which, and to make it the more ſimple and eaſy, we muſt conſider the natural Progreſſion 1 . 2 . 3 . *&c.* (from which the Triangulars proceed) as the Sums of a Series of Units 1 . 1 . 1 . 1 . *&c.* for the Sums of theſe continually from the beginning are 1 . 2 . 3 . 4 . *&c.*

Let us then begin with the Series of Units, and take the Series of their Sums, and the Sum of theſe Sums, and ſo on; and thus we ſhall have, from the moſt ſimple Original, the whole Orders of Polygonals of the Triangular Kind; and though, properly

ſpeaking, the Sums of the Series 1 . 2 . 3 . *&c.* are the firſt Order of Triangulars, yet it is convenient that we diſtinguiſh all the Series of the following Table by different Orders, calling the Series 1 . 1 . 1 . *&c.* the firſt Order, and 1 . 2 . 3 . *&c.* the ſecond; for they may be alſo call'd Triangular Numbers, becauſe what are more properly ſo proceed from them. It is true indeed, that by this Means the numbering of the Orders is different from the Method already laid down, but that will cauſe no Difficulty, becauſe this Way of numbering the Triangulars is uſed only with relation to the Rule we are now inveſtigating, and has this conſtant Connexion with the other, that the Number of the Order in this Method is always 2 more than that in the other Method; and beſides, by adding two Words we can ſave all Ambiguity; thus, when we ſpeak of any Order of Triangulars, for Example, the fourth Order, ſay the fourth Order from Units, and then all is clear; and if that is not added, you are to underſtand the Order number'd from the Series of Triangles (or Sums of the Series 1 . 2 . 3 . *&c.*) as in all the other Species the Orders are conſtantly number'd from their Polygons.

Table of Triangular Numbers from a Series of Units.

Orders									
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003

From the Conſtruction of this Table I make this uſeful *Obſervation.*

Every Term of every Order is equal to the Sum of the Collateral Term of the preceding Order, and the preceding Term of the ſame Order. Thus, 35 (the 5th Term of the 4th Order) is $=$ 15 (the Collateral or 5th Term of the preceding or 3d Order) $+$ 20 (the preceding or 4th Term of the ſame 4th Order.) The univerſal Truth of which Obſervation is manifeſt from the Conſtruction.

Theorem III.

The Series of Numbers of any Order of the Triangulars, deduced from a Series of Units, is the ſame Series as the Series of Collaterals, the Number of whoſe Place from the beginning is equal to the Number of the Order of the other. Thus the Collaterals in the 6th Place are the ſame as the Triangulars of the 6th Order, *viz.* 1 . 6 . 21. *&c.*

Demon. The Truth of this Theorem appears in the preceding Table, ſo far as it is carried; and the Conſtruction of the Table attentively conſider'd will make the Univerſality of it plain. But to remove all Difficulty, I ſhall prove it thus,

1°. Every Term of any Collateral Column is equal to the Sum of all the Terms of the preceding Column, from the ſame Order upwards. So in the Collaterals of the 4th Place, the Term $35=15+10+6+3+1$, the preceding Column. And the Univerſality of this is manifeſt from the Obſervation made above upon the Conſtruction of the Table; for the 1ſt Term in every Column is the ſame, *viz.* 1; then the 2d Term is the Sum of the firſt Term of the ſame Column, and the 2d Term of the preceding Column (by that Obſervation) *i. e.* the Sum of the 1ſt and 2d Terms of the preceding Column; the 3d Term is the Sum of the preceding, or 2d Term of the ſame Column (*viz.* the Sum of the 1ſt and 2d Terms of the preceding Column) and the correſponding or 3d Term of the preceding Column; and ſo on.

2°. From what is laſt ſhewn it is manifeſt, that the ſeveral Collateral Columns are alſo the Sums of Numbers taken continually from a Series of Units, which is the 1ſt Column;

lumn; and thence it appears plainly, that the perpendicular Columns at any Distance from the 1ſt Column of Units, muſt be *ad infinitum* the ſame as the tranſverſe Lines or Series of Numbers, at equal Diſtance from the Series of Units, which is the firſt Line.

Coroll. The Term in any Place of any Order of Triangulars (in the preceding Table) is equal to that Term whoſe Place is the Number of the Order of the former, and of ſuch an Order whoſe Number is the Place of the other. Thus, the 3d Term of the 5th Order is equal to the 5th Term of the 3d Order; and univerſally, the a Term of the b Order, is the ſame as the b Term of the a Order.

PROBLEM I.

To find the Triangular Number in any given Place, of any given Order (from Units.)

Rule. Let the Order be call'd a, and the Place b (or contrarily the Order b and the Place a) then take $n=a+b-2$, and carry on this Series $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. to $\frac{b}{a-1}$; the continual Product of all theſe Factors is the Number ſought.

Exam. To find the 5th ($=b$) Term of the 7th ($=a$) Order; then is $a+b-2$ ($=n$) $=7+5-2=10$; and the Number ſought is $1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3} \times \frac{7}{4} \times \frac{6}{5} \times \frac{5}{6} =$ 210.

Demon. 1°. The firſt Thing in order to the Demonſtration of this Rule, is to obſerve, That it is the very ſame Thing in effect as the Rule for finding the Coefficient of the $\bar{a}$ Term of a Binomial Power (or Power of a Binomial Root) whoſe Index is n, or $\overline{a+b-2}$, for which ſee *Book* III. *Chap.* II. So that what remains to be prov'd is this Correſpondence of Coefficients and Triangulars, *viz.* that the b Triangular (or Triangular in the b Place) of the $\bar{a}$ Order, is the ſame as the $\bar{a}$ Coefficient (or Coefficient of the $\bar{a}$ Term) of the $n=a+b-2$ Power of a Binomial Root. And to ſhew this let us,

2°. Compare the Table of Coefficients (*Book* III. *Chap.* II.) with this Table of Triangulars, and it's manifeſt they are the very ſame Numbers, only diſpoſed in another Manner. For it is plain, they are the ſame Numbers taken in their perpendicular Columns, as being produc'd the ſame Way from the Column of Units by continual Addition, *i. e.* what are there called ſimilar Coefficients are the ſame Numbers as what are here call'd Collaterals, being taken at equal Diſtance from the Beginning or Column of Units: The Difference being this, that in the Table of Triangulars the firſt Terms of every perpendicular Column ſtand in one Line, and ſo do the 2d Terms, and ſo on; but in the Table of Coefficients the firſt Term of the 2d Column ſtands in a Line with the 2d of the firſt Column, and ſo on; whence it's plain, that Coefficients in different Places, and different Powers, are the ſame Numbers under a different Name, with Triangulars in different Places and different Orders; and for their mutual Correſpondence let us conſider what is ſhewn, *Book* III. *Chap.* II. viz.

3°. The a Coefficient of the n Power is equal to the $n-a+2$ Coefficient of the ſame Power, reckoning from either Extreme; alſo that the $n-a+2$ Coefficient of the n Power is the ſame as the $n-a+2$ Term of the ſimilar Coefficients in the a Place of different Powers, *i. e.* (by what is before ſhewn) the Triangular in the a Place of the $n-a+2$ Order (for the different Places in the Column of ſimilar Coefficients anſwer to the different Orders of Triangulars; and different Places of the Coefficients of the ſame Powers anſwer to different Places in the ſame Order of Triangulars.) Now ſuppoſe

pose $n-a+2=b$, then is $n=a+b-2$; wherefore the a Coefficient, and the b Coefficient of the n Power are the same. Also the b Coefficient of the n Power is the b Term of the similar Coefficients in the a Place of different Powers, equal to the a Triangular of the b Order: But the a, or also the b Coefficient of the n Power is $1 \times \frac{n}{1} \times \frac{n-1}{2}$, &c. so $\frac{n-a+2=b}{a-1}$, which must therefore be the a Triangular of the b Order (or b Triangular of the a Order) as it is, according to the Rule, which is therefore good.

Another Rule *for solving the preceding* Problem.

For the sake of a particular Use of it, I give you here another Rule for the preceding *Problem*; which is this,

Let a express the given Order of Triangulars, and n the Place of the Term sought; then the continual Product of these Factors, $1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, *&c.* to $\frac{n+a-2}{a-1}$ is the Number sought.

Exam. To find the 5th Term of the 7th Order; it is, $210=1 \times \frac{5}{1} \times \frac{6}{2} \times \frac{7}{3} \times \frac{8}{4} \times \frac{9}{5} \times \frac{10}{6}$.

DEMON. In the Demonstration of the former Rule it is shewn, that the b Term of the a Order, is the a Coefficient of the $n=a+b-2$ Power; and if instead of b we put n, then the n Term of the a Order is the a Coefficient of the $a+n-2$ Power; which by the Rule of Coefficients is, $1 \times \frac{a+n-2}{1} \times \frac{a+n-2-1}{2} \times \frac{a+n-2-2}{3}$, *&c.* to $\frac{n}{a-1}$, which is the same as the preceding Rule for Triangulars (*i. e.* for the n Term of the a Order;) but this Series is the same in effect as the other Series $1 \times \frac{n}{1} \times \frac{n+1}{2}$, *&c.* $\frac{n+a-2}{a-1}$; for it is manifest that the Denominators are the same, and the Numerators also, only in a reverse Order (which makes no change in the Product;) for both the Series have the same Number of Terms, as the Series of Denominators does clearly shew; and for the Numerators, the first and last of them are the same Numbers in both, only the first in the one is the last in the other; and since their Progression is by a continual Difference of 1, it follows plainly, that they must be the same Numbers only in a reverse Order; consequently this Rule is good, since it is the same (only in a different Form) with the former, which is demonstrated to be good.

But observe, That this last Rule may also be demonstrated independently of the other, from the immediate Consideration of the Triangular Numbers, without any Comparison of them with Coefficients. *Thus*,

The n Term of the a Order is the a Term of the n Order (*Cor. Theor.* III.) and therefore it is the same Thing to which of these we apply the Rule; but for the present Demonstration we must take it, the a Term of the n Order; and then, I say, if the Rule is good in one Case, or for one Order of Triangulars, as the n Order, it will therefore be good in the next Case, or the $n+1$ Order, and consequently it is good in all superior Cases; and to prove this, first consider the annex'd Series, wherein, because 1 does not multiply, therefore I have omitted it in every Term as useless, except in the first Term, which is it self 1.

Tri-

	1st.	2d.	3d.	4th.	
Triangulars of the Order n, according to the Rule,	1.	$\frac{n}{1}$.	$\frac{n}{1}\times\frac{n+1}{2}$.	$\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$.	&c.
The Equivalents to these; (the Numerators being only in a different Order)	1.	$\frac{n}{1}$.	$\frac{n+1}{1}\times\frac{n}{2}$.	$\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n}{3}$.	&c.
Triangulars of the Order $n+1$ according to the same Rule.	1.	$\frac{n+1}{1}$.	$\frac{n+1}{1}\times\frac{n+2}{2}$.	$\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n+3}{3}$.	&c.

Now 'tis plain, that the Terms of the last Series are the Triangulars of the Order $n+1$, according to the Rule; and that they are truly the Numbers sought, upon Supposition that those of the former Series are the Triangulars of the Order n, I thus shew. By the Observation made upon the Construction of the Table of Triangulars, every Term of any Order is the Sum of the Collateral Term of the preceding, and the preceding Term of the given Order; but comparing the 2d and 3d Series (*i. e.* the Series for the Order n, and that for the Order $n+1$) it's plain that the last is compos'd according to that Property now mention'd; thus $1=1$, the first Terms; $1+\frac{n}{1}=\frac{n+1}{1}$ (*i. e.* the 2d Term of the Order $n+1$ equal to the 2d Term of the Order n, and 1st Term of the Order $n+1$;) $\frac{n+1}{1}\times\frac{n}{2}+\frac{n+1}{1}$ $\left(=\frac{n+1}{1}\times\overline{\frac{n}{2}+1}\right)=\frac{n+1}{1}\times\frac{n+2}{2}$. Again, $\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n}{3}+\frac{n+1}{1}\times\frac{n+2}{2}$ $\left(=\frac{n+1}{1}\times\frac{n+2}{2}\times\overline{\frac{n}{3}+1}\right)=\frac{n+1}{1}\times\frac{n+2}{2}\times\frac{n+3}{3}$, and so on; for it's manifest, that according as these two Series proceed, they must always have the same Connection, *viz.* that any Term of the last is the Sum of the Collateral Term of the former, and the preceding Term of the same last Series. Wherefore if the former is the Series of Triangulars of the Order n, the last must be that of the Order $n+1$.

But the Rule is true when apply'd to the first Order or Series of Units; for here $n=1$; and hence it's plain that the Numerator and Denominator in every Factor are equal, and therefore they are each equal to 1; hence every Term of the Series is 1.

Lastly, The Rule being good for the first Order, it must therefore, by what was first proved, be good for the second Order, and so for the third, and all the following for ever.

Observe, Lest any body should think the universal Connexion of the two Series, for the Order n and $n+1$, any thing obscure, I shall make this universal Demonstration of it, *viz.* That any Term of the last is the Sum of the Collateral Term of the former, and the preceding Term of the same last. Thus, any Term of the Series for the Order $n+1$ may be expressed $\frac{n+1}{1}\times\frac{n+2}{2}$, &c. to $\frac{n+a}{a}$ inclusive; and the preceding Term is therefore $\frac{n+1}{1}\times\frac{n+2}{2}$, &c. to $\frac{n+a}{a}$ exclusive; also the Collateral Term of the Series for the Order n is $\frac{n+1}{1}\times\frac{n+2}{2}$, &c. to $\frac{n}{a}$, inclusive: Now the Sum of

$n+1$.

$\frac{n+1}{1}\times\frac{n+2}{2}$ &c. and $\frac{n+1}{1}\times\frac{n+2}{2}$ &c. $\times\frac{n}{a}$ is $=\frac{n+1}{1}\times\frac{n+2}{2}$ &c. $\times\overline{1+\frac{n}{a}}=\frac{n+1}{1}\times\frac{n+2}{2}$ &c. $\frac{n+a}{a}$, the Thing to be proved.

SCHOL. As this 2d Rule has been demonstrated independently of the Rule for Coefficients; so the Rule of Coefficients may be also demonstrated, by Means of this Rule, for Triangulars, with the Correspondence betwixt the two, as above explain'd. Thus, the n Term of the a Order of Triangulars, being $1\times\frac{n}{1}\times\frac{n+1}{2}$ &c. $\times\frac{n+a-2}{a-1}$; let us only invert the Order of the Numerators, which does not alter the Value of the Total Product, and it is $1\times\frac{n+a-2}{1}\times\frac{n+a-2-1}{2}$ &c. $\times\frac{n}{a-1}$. Again, Take m $=n+a-2$, and the Series is $1\times\frac{m}{1}\times\frac{m-1}{2}$ &c. $\times\frac{m-a+2}{a-1}$, which is the Rule for the a Coefficient of the m Power, as it ought to be; since it is shewn that the n Term of the a Order of Triangulars, is the a Coefficient of the $n+a-2$ Power, *i. e.* of the m Power.

Hence we have also a new Rule for Coefficients, which is this; Let n, or a, be the Place of the Coefficient, and $a+n-2$, the Index of the Power; then is $1\times\frac{n}{1}\times\frac{n+1}{2}$ &c. $\times\frac{n+a-2}{a-1}$, the Coefficient; for this is the a Term of the n Order of Triangulars; which is equal to the n Coefficient of the $a+n-2$ Power, as already shewn. Also the n Coefficient of the $a+n-2$ Power, is equal to the a Coefficient of the $a+n-2$ Power; for if you call $a+n-2=b$, then the n Coefficent of the b Power is also the $b-n+2$ Coefficient of the b Power (as has been shewn) *that is*, the $a+n-2-n+2$ $(=a)$ Coefficient of the $a+n-2$ Power.

In the last Place we shall set before us, in one View, these two Rules, as they relate both to Coefficients and Triangulars.

$1\times\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ &c. $\times\frac{n-a+2=b}{a-1}=$ a, or also the $n-a+2$ $(=b)$ Coefficient of the n $(=a+b-2)$ Power. a Triangular of the b Order, or b Triangular of the a Order.

$1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$ &c. $\times\frac{n+a-2=b}{a-1}=$ n or a Coefficient of the b $(=n+a-2)$ Power. n or a Triangular of the a or n Order.

PROBLEM II.

To find the Polygonal Number in any Place, of any Order, and of any Species.

Rule. Find, by the last Problem, the Polygonal of the given, and also of the preceding Place, of the given Order of the first or triangular Species. Take the Number of the given Species, less 1; by which multiply the Polygonal found of the preceding Place; to the Product add the Polygonal found of the given Place; the Sum is the Number sought. But here *observe*, That in every Species, except the first, the Orders are number'd from the Polygons, or Sums of the Arithmetical Series, whence they proceed: Whereas in the first Species they are number'd from the Series of Units; so that their Number is always more by 2 than that of the other Orders, at the same Distance from the Polygons. Wherefore in finding the preceding Polygonal of the first Species, and of the given Order, from Polygons, add 2 to the Number of

of the given Order, and find the Polygonal for that Order, according to the preceding Rules, which will give a Polygonal at the ſame Diſtance from the firſt and ſimple Polygons as the given Order is.

Exam. To find the 4th Term of the 3d Order of the 5th Species; I find the 3d and 4th Terms of the 5th Order of the 1ſt Species, which are 15, 35; then the Number of the given Species is 5, and $15\times4=60$; to which add 35, the Sum is 95, the 4th Term of the 3d Order of the 5th Species. (*See the Table.*)

DEMON. The Reaſon of this Rule is manifeſt from *Coroll.* to *Theor.* II. and needs no farther Explication.

SCHOLIUM, *relating to* Problem I *and* II.

From theſe general Rules of *Probl.* I, and II. we may eaſily deduce particular Rules for particular Series: I ſhall apply them to two Caſes, by which all others will be eaſily underſtood.

1°. To find the Sum of the Series of Triangles to any Number (n) of Terms, *i. e.* to find the n Term of the Series of Triangular Pyramids or Polygonals of the 1ſt Species, and 4th Order from the Series of Units (which is the 2d Order from the ſimple Polygons.) The Rule is,

To twice the Side or Place of the Term ſought, add its Cube, and thrice its Square; the 6th Part of the Sum is the Term ſought, *viz.* $\frac{n^3+3n^2+2n}{6}$; for by the general Rule of *Problem* I. this Term ſought is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$.

2°. To find the Sum of Polygons of the 2d Species, or Squares, to any Number of Terms, *i. e.* to find any Term of the Series of ſquare Pyramids, or Polygonals of the 2d Species and 4th Order from the Series of Units (which is the 2d Order from the ſimple Polygons.) The *Rule* is this:

To the Number of Terms added, or Place of the Term ſought, add thrice its Square, and alſo double its Cube; the 6th Part of the Sum is the Number ſought. Thus, if the given Number of Terms is n, the Number ſought is $\frac{2n^3+3n^2+n}{6}$: The *Inveſtigation* of which is this; the n Term of the 4th Order of Triangulars (from the Series of Units) is $=\frac{n^3+3n^2+2n}{6}$ (by the laſt) and the preceding, or $n-1$ Term is $1\times\frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}=\frac{n^3-n}{6}$; but, by *Coroll. Theor.* II. the n Term of the 2d Species is the Sum of the n and $n-1$ Terms of the 1ſt or Triangular Species; *i. e.* $\frac{n^3+3n^2+2n}{6}+\frac{n^3-n}{6}=\frac{2n^3+3n^2+n}{6}$.

THEOREM IV.

Take the Series of ſimple Polygons of any Species, after the 1ſt, to any Number of Terms; and take the Series of Triangulars of any Order after the 1ſt (numbering the Orders of theſe from the Series of Units) to the ſame Number of Terms; place theſe reverſely under the other, and multiply the correſponding Terms (as they are plac'd) of the one into thoſe of the other; the Sum of the Products is equal to a Term ſtanding in the Place expreſs'd by the given Number of Terms, of the Order expreſs'd by 1 more than the given Order of Triangulars, and of the given Species of Polygonals.

Exam.

Exam. The Sexangulars to the 4th Term of the 1st Order, are 1 . 6 . 15 . 28; the Triangulars of the 5th Order (from Units) are 1 . 5 . 15 . 35; which placed under the other, and multiplied as in the Margin, produces 228; which is the 4th Sexangular of the 6th Order from the simple Hexagon (which is here the 1st Order) for this is 1 . 6 . 15 . 28; the 2d Order is 1 . 7 . 22 . 50; the 3d Order is 1 . 8 . 30 . 80; the 4th Order is 1 . 9 . 39 . 119; the 5th Order is 1 . 10 . 49 . 168; the 6th Order is 1 . 11 . 60 . 228.

1 . 6 . 15 . 28
35 . 15 . 5 . 1

35 + 90 + 75 + 28 = 228

Demon. The Reason of this is plain from the following Table; wherein, if 1 . *a*. *b* . *c* . *d* . &c. represent the Polygons of any Species, the several Orders of Sums proceeding from these, are evidently as in this Table.

1	1	1	1
a	$1+a$	$2+a$	$3+a$
b	$1+a+b$	$3+2a+b$	$6+3a+b$
c	$1+a+b+c$	$4+3a+2b+c$	$10+6a+3b+c$
d	$1+a+b+c+d$	$5+4a+3b+2c+d$	$15+10a+6b+3c+d$
&c.	&c.	&c.	&c.

And it is plain also, that the Terms or Sums in each Order are according to the *Theorem*; because in the 2d Order they are the Sums of the Series of the first Order, multiplied by a Series of Units (which is the 1st Order in Triangulars) then the Multipliers in all that follow, are manifestly the Sums of the preceding continually, from the Series of Units.

Coroll. Hence we learn a new Practice for finding the Polygonal, in any Place, of any Order and Species after the 1st, *viz.* by having the Series of the 1st Order of any Species (after the 1st) and the Series of the Triangulars of the Order 1 less than the other: But this not being so easy a Practice as that in the preceding *Problem*, I have chosen to express the Rule in the Manner of a *Theorem*, regarding it only in general, as a Connection discovered betwixt the Triangulars and the other Species of Polygonals.

Theorem V.

If we take the Progression, 1 . 2 . 3 . 4 &c. and the Series of Triangles, which are the Sums of the former, 1 . 3 . 6 . 10 . 15 &c. then take the Series of Ratios of the several Terms of the 1st Series, comparing each Term to the following, in a continued Order, as, 1 : 2, 2 : 3, 3 : 4, &c. Also take the Series of Ratios of the 2d Series, beginning at the 2d Term, and proceeding discontinuedly; *thus*, 3 : 6, 10 : 15, &c. These two Series of Ratios are the same; thus, 1 : 2 : : 3 : 6, 2 : 3 : : 10 : 15, and so on.

Demon. I have in the Margin placed the 2 Series according to the proposed Correspondence of their Ratios; and so far as it is carried, the Truth of the *Theorem* is plain. But to shew the Reason of it, and that it must be so for ever; in the first Place *observe*, that the Antecedents in the several Ratios of the 1st Series (1 . 2 . 3 . &c.) express the Places of these Terms from the Beginning; and the several Antecedents of the Ratios taken in the 2d Series, stand in the several even Places of the Series, *i. e.* in the 2d, 4th, 6th, *&c.* Places; but the Series of even Numbers, 2 . 4 . 6 &c. are the Doubles of the respective Terms of the natural Progression, 1 : 2 : 3 : &c. which being all Antecedents of the Ratios taken in the 1st Series, it follows, that the Antecedents of the several

1 : 2, 2 : 3, 3 : 4, &c.
3 : 6, 10 : 15, 21 : 28, &c.

feveral Ratios in the 2d Series are in fuch Places of that Series as are expreffed by double the Antecedent of the correfpondent Ratio (number'd from the Beginning of the Series of Ratios) in the 1ft Series; *thus*, 3 : 4 is the 3d Ratio of the 1ft Series; and the 3d Ratio of the 2d Series, is 21 : 28, whofe Antecedent, 21, ftands in the 6th ($=2\times 3$) Place of the Series. Now to fhew that thefe correfpondent Ratios are equal, take any two Terms adjacent in the 1ft Series; they may be expreffed $n : n+1$, which make the nth Ratio in the Order of Ratios, as they are taken out of the 1ft Series: And by what is laft fhewn, the Antecedent of the nth Ratio of the 2d Series, according to the Manner of taking them there, ftands in the $2n$ Place of that Series; and confequently it is the Sum of the 1ft Series to the $2n$th Term; which, by the Rule of Progreffions, is $\overline{2n+1}\times\frac{2n}{2}=\overline{2n+1}\times n=2n^2+n$; and the Confequent or next greater Term in the Series of Sums muft be this Sum, with the following Term of the 1ft Series, which is $2n+1$; fo the Confequent is $2n^2+n+\overline{2n+1}=2n^2+3n+1$. Then *laftly*, $n : n+1 :: 2n^2+n : 2n^2+3n+1$; becaufe the Product of Extremes and Means are equal, *viz.* $2n^3+3n^2+n$.

THEOREM VI.

Take the natural Series, 1, 2, 3, &c. alfo the Series of its Sum, or Series of Triangles, 1 . 3 . 6 . &c. the Series of Ratios proceeding from the Comparifon of every Term of the 1ft Series to the 2d from it, or next but one, as 1 : 3, 2 : 4, 3 : 5, &c. are the fame as thofe, which proceed from the Comparifon of every Term of the 2d Series to the next, as 1 : 3, 3 : 6, 6 : 13, &c.

Series.
1 . 2 . 3 . 4 . 5 . 6.
1 . 3 . 6 . 10 . 15 . 21.

Ratios.
1:3, 2:4, 3: 5, 4: 6.
1:3, 3:6, 6:10, 10:15.

DEMON. Let n be any Term of the 1ft Series, and $n+2$ the 2d above it; then is the n Term of the 2d Series $\frac{\overline{n+1}\times n}{2}$, and the next Term above it, or the $n+1$ Term, is $\frac{\overline{n+2}\times\overline{n+1}}{2}$; but it's plain, that $n : n+2 :: n\times\overline{n+1} : \overline{n+2}\times\overline{n+1}$; or as $\frac{n\times\overline{n+1}}{2} : \frac{\overline{n+2}\times\overline{n+1}}{2}$; the Thing to be proved.

THEOREM VII.

Take any three adjacent Triangles, and betwixt the leffer and the middle one, place the Number next leffer (*viz.* by 1) than that middle one; and thefe four are Geometrically Proportional: Thus for *Example*, 6 . 10 . 15, are three adjacent Triangles, and 6 : 9 :: 10 : 15.

DEMON. Take three Triangles ftanding in the $n-1$, n, and $n+1$, Places; they are $\frac{n\times\overline{n-1}}{2}$; $\frac{\overline{n+1}\times n}{2}$; $\frac{\overline{n+2}\times\overline{n+1}}{2}$; from the middle one $\frac{\overline{n+1}\times n}{2}$ take 1, the Remainder is $\frac{\overline{n+1}\times\overline{n-2}}{2}$; and $\frac{n\times\overline{n-1}}{2} : \frac{\overline{n+1}\times\overline{n-2}}{2} :: \frac{\overline{n+1}\times n}{2} : \frac{\overline{n+2}\times\overline{n+1}}{2}$, as will appear from the equal Product of Extremes and Means; and to do this more eafily, becaufe the Denominators are all equal, we may caft them all out; and then alfo obferve, that $\overline{n+1}\times n : \overline{n+2}\times\overline{n+1} :: n : n+2$; wherefore we need only try the Proportionality of thefe, $n\times\overline{n-1} : \overline{n+1}\times\overline{n-2} :: n : n+2$, *i. e.* $nn-n : nn+n-2 :: n : n+2$, in which $\overline{nn-n}\times\overline{n+2}=\overline{nn+n-2}\times n$; therefore the 4 are ::, *i. e.* the Numbers propofed are ::.

Coroll. The Product of any two Triangles, betwixt which there lies but one other Triangle, is an Oblong, whose greater Side is that interjacent Triangle.

Lemma.

Take the natural Progression, 1, 2 3, &c. and after the 1st Term 1, take the Sum of every two successive Terms, *thus*, 1, 2+3, 4+5, 6+7 &c. you have hereby a Series of Numbers, $\div l$ with the common Difference, 4. Thus the preceding Series is 1 . 5 . 9 . 13 . &c.

Demon. The Reason is plain from this, That every Term in the natural Series, 1 . 2 . 3 . &c. exceeding the preceding, by 1, betwixt any Term and the next but 1 (or the 2d after it) the Difference is 2; consequently the Difference of the Sum of any two adjacent Terms, and the Sum of the next two adjacent Terms, must be 4; therefore the Series of these Sums are in a constant Difference of 4; which is also the Difference of 1, and the 1st Sum 2+3.

Theorem VIII.

Every *Hexagon* is also a *Triangle*; and particularly, all the Triangles in odd Places, as the 1st, 3d, 5th, &c. make the complete Series of Hexagons: As here,

	1 .	2 .	3 .	4 .	5 .	6 .	7 .	8 .	9	
Triangles	1 .	3 .	6 .	10 .	15 .	21 .	28 .	36 .	45	&c.
Hexagons	1 :		6 :		15 :		28 :		45	
	1 .		5 .		9 .		13 .		17	

Demon. Hexagons are the Sums of a Series $\div l$, whose first Term is 1, and the common Difference 4 (as 1 . 5 . 9 &c.) and the Triangles are the Sums of the natural Series (1 . 2 . 3 . &c.) but taking this last Series in the Manner mentioned in the preceding *Lemma*, viz. 1 : 2+3 : 4+5 : &c. we have a Series beginning with 1, and proceeding with the Difference 4; consequently the Sums of this Series are Hexagons; but it's plain, that they are also Sums of the natural Progression taken to every odd Number of Terms; for they are 1, 1+2+3, 1+2+3+4+5, &c. and consequently they are all the Triangles in odd Places.

Otherwise *thus*; If you number the odd Places of any Series by themselves, and compare the Number in any odd Place, with the Place of that Term, as it's number'd with all the Terms of the Series; then if to the Number of the Place, in which any odd Term stands in the whole Series, be added 1, the Half of the Sum expresses what Place it stands in among the odd Terms number'd by themselves; *thus*, The 9th Term, in the Whole, is the 5th $\left(=\frac{9+1}{2}\right)$ Term of the odd Places numbered by themselves; the Reason of which is obvious. Again, Any odd Number may be expressed $2n+1$, and if the Sum of the natural Series is taken to the $2n+1$ Term, it is $\overline{2n+2}\times\frac{2n+1}{2}=\frac{4nn+6n+2}{2}=2nn+3n+1$, which is a Triangle. Then to the last Term (or Number of Terms) added in this Sum, *viz.* to $2n+1$, add 1; the Sum is $2n+2$, whose Half is $n+1$; which, by what's shewn, is the Place of the Number $2n+1$, among the odd Places of the natural Series, number'd by themselves; wherefore, find the $n+1$ Hexagon, and it is $2nn+3n+1$, which is the Triangle already found in the $2n+1$ Place: For Hexagons proceed from a Series $\div l$, whose Difference is 4; where-

wherefore the $n+1$ Term of that Series is $1+4n$; and hence the Sum of the Series to the $n+1$ Term is $\overline{2+4n}\times\frac{n+1}{2}=\frac{4nn+6n+2}{2}=2n^2+3n+1$.

THEOREM IX.

If the Sums of the odd Series, 1 . 3 . 5 . 7 &c. are taken continually from the Beginning, they are the Squares of the natural Progression, 1 . 2 . 3 . 4 . &c. Or thus; Every Square Number is the Sum of the Terms of the odd Series taken from 1, to a Number of Terms equal to the Root of that Square.

Odd Series ————	1	3	5	7	9	11	13	15	&c.
Their Sums ————	1	4	9	16	25	36	49	64	&c.
Square Roots of the Sums	1	2	3	4	5	6	7	8	&c.

DEMON. The Truth of this Proposition you see so far as the Series are carried; and that it will be so for ever we have already demonstrated, in *Cor.* 4. *Probl.* V. *Ch.* II, *&c.* where it is shewn, that the Sum of the odd Series, 1 . 3 . 5 . &c. is the Square of the Number of Terms.

But there is another more natural Demonstration of this Truth, deduced from the Consideration of square Numbers, and their Composition. Thus;

1°. Take the natural Progression, 1 . 2 . 3 . &c. and the odd Series, 1 . 3 . 5 . &c. the several Terms of the natural Series, 1 . 2 . 3 . &c. express the Number of Terms from 1 to any Term of the odd Series, or to any Term of the Series of their Sums; but from the Nature of Progressions, and particularly of this odd Series, any Term of it is equal to the Sum of 1 (the lesser Extreme) and 2 (the common Difference) multiplied by the preceding Term of the natural Progression (which is the Number of Terms less 1.)

2°. The Difference of any two square Numbers, whose Roots differ by 1 (and such are every two adjacent Terms in the Series, 1 . 2 . 3 . &c.) is equal to the Sum of 1, and double the lesser Root; thus, $\overline{a+1}^2=a^2+2a+1$; so that a^2, and $\overline{a+1}^2$, differ by $2a+1$. Hence,

3°. If the Sum of the odd Series, carried to any Number of Terms, is the Square of the Number of Terms (*i. e.* of the correspondent Term of the natural Progression, 1 . 2 . 3 . &c.) so will it be if carried to one Term more; because that next odd Term is equal to the Sum of 1, and double the preceding Root (or Term of the Series, 1 . 2 . 3 . &c.) which is also the Difference of two Squares, whose Roots differ by 1; as it is in the present Case. But we see the Truth proposed as far as we have carried the Series; therefore it must go on so for ever.

COROLLARIES.

1*st.* The Difference of any two integral Squares is equal to some one, or the Sum of some two or more Terms of the odd Series: More particularly, it is equal to the Sum of all the Terms of the odd Series comprehended betwixt that Term (inclusive) whose Place in the Series is the Root of the greater Square, and that Term (exclusive) whose Place is the Root of the lesser Square, *i. e.* all the Terms from that one (inclusive) which stands over the greater Square, and that one (exclusive) which stands over the lesser. So if n represents the Place of any Term in the odd Series, and m the Place of any lesser Term; then are n, $m-1$, the Roots of two Squares, which differ by the Sum of all the Terms comprehended betwixt these Extremes, including both. *Exqm.* $49-9=7+9+11+13$. Hence,

2*d.* 2 cannot be the Difference of any two Squares; becauſe it is not any Term, nor the Sum of any Terms of the odd Series, the two leaſt being $1+3=4$.

3*d.* If the Difference of two Squares is equal to the Sum of all the Terms of the odd Series, from 1 to any aſſigned Term; the greater of theſe Squares cannot be that correſponding to the aſſigned Term, *i. e.* it cannot have for its Root the Place of (or Number of Terms from the Beginning to) that Term; becauſe no leſſer Square can differ from that greater one, by the Sum of all the odd Series, from 1 to that greater. Hence again,

4*th.* 1 cannot be the Difference of two Squares, nor 4; becauſe $1+3$ cannot be the Difference of two Squares, whereof the greater correſponds to 3; nor is 4 any Term of the odd Series, or the Sum of any two or more Terms of the odd Series, other than $1+3$.

5*th.* Every odd Number above 1, and the Sum of any Number of Terms adjacent in the odd Series, whereof the leſſer is greater than 1, is the Difference of ſome two Squares, whoſe Roots are found as in the firſt *Corollary*.

$$\begin{array}{l} 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \\ 11 \,.\, 9 \,.\, 7 \,.\, 5 \,.\, 3 \,.\, 1 \\ \hline 11+18+21+20+15+6= \\ 1+4+9+16+25+36= \\ 91. \end{array}$$

6*th.* If we take the natural Series, 1 . 2 . 3 . &c. to any Number of Terms, and under it ſet the Series of odd Numbers, 1 . 3 . 5 . &c. in a reverſe Order; then multiply each Term of the one into the correſponding of the other; the Sum of theſe Products is equal to the Sum of the Squares of all theſe Terms of the Series, 1 . 2 . 3 . &c.

Schol. That neither 1 . 2, or 4, can be the Difference of any two Squares, may be eaſily ſhewn otherwiſe; thus, n^2 and $\overline{n+1}^2$ $(=n^2+2n+1)$ differ by $2n+1$, which, it's plain, can neither be 1 . 2, nor 4; and the leaſt it can be, is 3, *viz.* when $n=1$. If we take two Roots differing more than 1, as n, $n+d$, their Squares differ by $2nd+dd$; which, it's manifeſt, exceeds 4; for dd is here, at leaſt, 4; d being greater than 1.

A particular Uſe and Application of ſome of theſe Corollaries you'll find afterwards.

Theorem X.

Take the Series of Triangles, 1 . 3 . 6 . 10 . &c. Then take the Sum of every two adjacent Terms continuedly, thus, $1+3$, $3+6$, &c. The Sums are the Series of Squares of the natural Progreſſion after 1; as in the Margin.

1	2	3	4	5	6	*Nat. Series*
1	3	6	10	15	21	*Triangle*
1	4	9	16	25	36	*Squares*

Demon. This follows from the laſt, compared with *Cor.* 1. *Theor.* II. For the Sums of the Series, 1 . 3 . 5 . &c. are the Squares of the natural Series, 1 . 2 . 3 . &c. by the laſt, and by *Cor.* 1. *Theor.* II. the Sum of every two adjacent Triangles is the Term of the Quadrangles (or Squares) collateral with the greater of theſe Triangles.

But I ſhall alſo demonſtrate this *Theorem* otherwiſe; thus, Triangles are the Sums of the natural Series, 1 . 2 . 3 . &c. the Sums of which, to n, and $n+1$ Terms, are $\frac{nn+n}{2}$, and $\frac{nn+3n+2}{2}$, which are therefore the two adjacent Triangles in the n and $n+1$ Places; but their Sum is $\frac{2nn+4n+2}{2}=nn+2n+1=\overline{n+1}^2$.

THEOREM XI.

Some Triangles are alſo Squares: And if beginning with the Numbers, 2 : 3, you make another Couplet out of them, thus, Take the Sum of theſe two for one Term (*viz.* $2+3=5$) then to that Sum add the leſſer Term of the ſame Couplet (*viz.* 2.) and make this Sum ($5+2=7$) the other Term: Then out of this laſt Couplet, 5 : 7, make another Couplet in the ſame Manner as before; which will be 12 : 17 (*viz.* $5+7=12$, and $12+5=17$.) And go on in this Manner for ever. Again, Take the Squares of both Terms of each of theſe Couplets; the Products of the Squares of the two Terms of every Couplet, are all Numbers which are both Squares and Triangles: As in the following Scheme; where 36 is a Square, whoſe Root is 6; and it is a Triangle, the Sum of the 1ſt 8 Terms of the natural Series.

	2 : 3,	5 : 7,	12 : 17,	&c.
Squares	4 : 9,	25 : 49,	144 : 289,	&c.
Products	36 .	1225 .	41616 .	&c.

DEMON. 1°. The Numbers here produced are Squares; becauſe they are the Products of two Squares (*Theor.* B. III. *Ch.* I.)

2°. They are alſo Triangles, or the Sums of the natural Series to a certain Number of Terms; which I prove by theſe Steps.

(1.) If any two Numbers, a, b, are ſuch, that $b=2a+1$, or $2a-1$; then is ab the Sum of the natural Series, from 1 to $2a$, in the Caſe of $b=2a+1$; or from 1 to b, if $b=2a-1$: For if $b=2a+1$, then are $2a$, and b, two adjacent Terms in the natural Series; and the Sum of the Series to $2a$, is ab; becauſe $b=2a+1$ the Sum of Extremes, and a is the Half of $2a$, the Number of Terms: Again, If $b=2a-1$, then alſo are b, $2a$, two adjacent Terms in the natural Series, and the Sum to b is ab; for $2a$ is the Sum of the Extremes, whoſe Half, a, multiplied into the Number of Terms, b, gives the Sum: What remains to be ſhewn is, that the Squares of each Couplet (as 4 : 9, or 25 : 49) are ſuch Numbers as a, b are here ſuppoſed to be; which is made out thus:

(2.) If any two Numbers, a, b, (which now repreſent any of the Couplets of the firſt Line) are ſuch, that $2aa+1$ is $=bb$; then let a, b, again repreſent the next Couplet; and here it will be $2aa-1=bb$: Or, if it was $2aa-1=bb$ in the former, it will be $2aa+1=bb$ in this. Which I thus demonſtrate: The Terms of one Couplet being called a, b; the Terms of the next, made out of the former according to the Propoſition, are $a+b$, and $2a+b$; and I ſay, that $2\times\overline{a+b}^2-1$, or $2\times\overline{a+b}^2+1$ is $=\overline{2a+b}^2$; according as $2aa+1$, or $2aa-1$ is $=bb$ in the former Couplet: For $2\times\overline{a+b}^2-1=2aa+4ab+2bb-1=6aa+4ab+1$ (by ſubſtituting $2aa+1$ for bb) alſo $\overline{2a+b}^2=4aa+4ab+bb=6aa+4ab+1$ (by the ſame Subſtitution.) Hence it is plain, that $2\times\overline{a+b}^2-1=\overline{2a+b}^2$. Again, $2\times\overline{a+b}^2+1=2aa+4ab+2bb+1$ $=6aa+4ab-1$ (by ſubſtituting $2aa-1$ for bb) and $\overline{2a+b}^2=4aa+4ab+bb$ $=6aa+4ab-1$ (by the ſame Subſtitution) whence $2\times\overline{a+b}^2+1=\overline{2a+b}^2$.

(3.) The 1ſt Couplet, 2, 3, is ſuch, that putting $2=a$ and $3=b$, then is $2aa+1=bb$: And therefore in the next it is $2aa-1=bb$; and ſo on alternately, by what's ſhewn in the laſt Step.

Laſtly, From all theſe Premiſes it follows, That the Product of the Squares of each Couplet is a Triangle; for by the firſt Step it is ſhewn, That if $2a+1$, or $2a-1$ is

is $=b$, then is ab a Triangle. Or, which is the ſame Thing, ſubſtitute aa, bb, for a, b; and ſince it is every where $2aa+1$, or $2aa-1=bb$ (by the 2d and 3d Steps) therefore $aa\times bb$ is a Triangle (by the 1ſt Step.)

SCHOL. As the Truth contained in this Theorem is a plain and direct Solution of this Problem, *viz. To find a Square Number, which is alſo the Sum of a certain Number of Terms of the natural Series* 1 . 2 . 3 . &c. So we have here alſo learned the Solutions of the following Problems.

COROLL. 1. (*Problem*) To find a Number of Terms, to which if the natural Series is carried, the Sum is a Square Number.

Rule. Take any of the Couplets of Squares of the preceding Scheme, as 4 : 9, or 25 : 49. Double the leſſer Square, and if this Double is leſs than the other Square, it is the Number ſought; but if it's greater, then the other Square is the Number ſought. Thus, 8 ($=2\times4$) is ſuch a Number as is required, becauſe 8 is leſs than 9: Again, 49 is ſuch a Number, becauſe 50 ($=2\times25$) is greater than 49; and ſo on through all theſe Couplets, double of the leſſer Square, and the greater Square, are alternately Solutions of this Problem.

The Reaſon of this *Rule* is plainly contained in the Demonſtration of the preceding Theorem; for it's ſhewn that the Product of any of theſe Couplets of Squares, as $a^2\times b^2$, is a Triangle, or the Sum of a certain Number of Terms of the natural Series; which was deduced from the Suppoſition that $b^2=2\times a^2+1$, or $2\times a^2-1$; whereby it's plain, that b^2 and $2a^2$ differ by 1, and conſequently ſtand next together in the natural Series, $2a^2$ being leſs than b^2 in the firſt Caſe, but greater in the other; whence the Sum of the Series to $2a^2$ Terms in the one Caſe, and to b^2 Terms in the other, is, by the Rules of Progreſſion, $a^2\times b^2$.

COROLL. 2. (*Probl.*) To find two Squares (or two Numbers whoſe Squares are) ſuch, that the greater Square, and the double of the leſſer, differ by 1.

Rule. The Solution and Reaſon of this Problem is plainly contained in the Theorem; for the ſeveral Couplets of Squares (or their Roots) whoſe Products are both Squares and Triangles, ſolve this Problem, becauſe it's ſhewn, that $a^2\times b^2$ is a Triangle, for this very Reaſon, that $2aa+1$, or $2aa-1$ is $=b^2$.

COROLL. 3. (*Probl.*) To find a Number, which added to its Square, the half Sum is alſo a Square.

Rule. Any Number which ſolves the Problem in *Cor.* 1. ſolves this alſo; for there it is ſhewn, that if $a^2\times b^2$ is a Triangle, it's the Sum of $2a^2$ Terms of the natural Series, ſuppoſing $b^2=2\times a^2+1$; or the Sum of b^2 Terms, if $b^2=2a^2-1$; but the Sum of the natural Series carried to any Number, as n Terms, is $\frac{nn+n}{2}$, which in the preſent Caſe is alſo a Square.

§ 3. *Of Priſmatick Numbers* (ſee the Tables after the 9th *Defin.* § 1.)

THEOREM XII.

THE Collaterals in any Place, of any the ſame Degree and Order of Priſmaticks, are in Arithmetical Progreſſion.

DEMON. 1°. It's ſo in the Collaterals of every Degree of the 1ſt Order, becauſe they are, by the Conſtruction, Products of the Collaterals of the ſame Place of the 1ſt Order of Polygonals, multiplied by ſuch a Power of the Side or Place whoſe Index is the Degree of the Priſmaticks. But the Collateral Polygons are in Arithmetical Progreſſion; and any ſuch Progreſſion being equally multiplied, the Products are alſo $\div l$.

2°. Since

2°. Since the Collaterals of the 1ſt Order are $\div l$, it follows from *Lemma* 1. that it is ſo in all the Orders of Sums proceeding from each of theſe Degrees.

Theorem XIII.

The Priſmatick in any Place of the 1ſt, or Triangular Species, and 1ſt Order of any Degree, is equal to the half Sum of theſe two Powers of the given Side or Place, whoſe Indexes are the given Degree $+1$, and $+2$; thus in the 8th Place of the 3d Degree, it is the $\frac{1}{2}$ Sum of the 4th and 5th Powers of 8.

Degrees	1	3	6	10	*Triangles*
1ſt	1	6	18	40	*Triangular Priſmaticks of the 1ſt Order.*
2d	1	12	54	160	
3d	1	24	162	640	
4th	1	48	486	2560	

Demon. Let n be the Side or Place of any Triangle, and the Triangle it ſelf is $\frac{nn+n}{2}$; alſo by the Conſtruction of the Priſmaticks, that in the n Place of the 1ſt Degree, and 1ſt Order is $\frac{nn+n}{2}\times n=\frac{n^3+n^2}{2}$. In the n Place, 2d Degree and 1ſt Order, it is $\frac{n^3+n^2}{2}\times n=\frac{n^4+n^3}{2}$; and ſo on. *Univerſally*, in the n Place of the 1ſt Order and m Degree it is $\frac{n^{m+2}+n^{m+1}}{2}$. Of which take Examples in the annex'd Scheme.

Theorem XIV.

Take the Series of Priſmaticks of the 1ſt Order of any Species, and any Degree, to any Number of Terms; alſo, take the Series of Triangulars of any Order [numbering from the Series of Units] to the ſame Number of Terms; place this Series under the other, in a reverſe Order, and multiply the correſponding Terms together, the Sum of the Products is the Priſmatick of the given Species and Degree, which ſtands in the Place expreſſed by the given Number of Terms, and of the Order expreſſed by 1 more than the given Order of Triangulars; or it is the Sum of the Series of Priſmaticks of the given Species, Degree and Number of Terms, and of the ſame Order as that of the Triangulars.

$$\begin{array}{c} 1 : 12 : 54 \\ 6 : 3 : 1 \\ \hline 6+36+54=96 \end{array}$$

Example. The Priſmaticks of the 1ſt Order, 2d Degree, 1ſt Species, to 3 Terms, are 1, 12, 54; the Triangulars of the 3d Order (from Units, which is the Order of ſimple Triangles) are 1 . 3 . 6; and theſe multiplied reverſely into the other, produce 96, the 3d Term of the Priſmaticks of the 1ſt Species, 2d Degree and 4th Order, as you'll find by carrying on the Sums; for theſe of the 1ſt Order being 1 . 12 . 54, of the 2d Order they are 1 : 13 : 67; of the 3d, 1 . 14 . 81; and of the 4th, 1 . 15 . 96.

Demon. The Reaſon of this is the ſame, as what has been explained in *Scholium* to *Probl.* 1. for finding the Polygonal in any Place of any Species, and any Order after the 1ſt; as you'll eaſily perceive by ſuppoſing the Expreſſions there aſſumed, *viz.* 1 . *a* . *b* . *c* . *d*, &c. to repreſent the Series of Priſmaticks of the 1ſt Order of any Species and Degree; for ſince the different Orders are the continual Sums of the preceding in Priſmaticks, the ſame way as in Polygonals, the Concluſion muſt be the ſame too.

Theo-

THEOREM XV.

The Difference in any Column of Collateral Prismaticks of the 1st Order, and of any Degree, is equal to the Product of the precedent Triangle by such a Power of the Side of the given Collaterals, whose Index is the given Degree of the Prismaticks. Or also, it is equal to the half Difference of these two Powers of the Number expressing the given Place, whose Indexes are the given Degree more 1 and more 2.

Exam. The common Difference in the Collateral Prismaticks in the 8th Place, 1st Order and 4th Degree, is equal either, 1°. To the Product of the 7th Triangle multiplied by the 4th Power of 8. Or, 2°. To the half Difference of the 5th and 6th Powers of the Number 8.

DEMON. 1°. The Difference in any Column of Collateral Polygons, is the precedent Triangle (*Theor.* II.) and the corresponding Collateral Prismaticks of the 1st Order, and of any Degree, are the Products of these Collateral Polygons, by such a Power of the Side or Place, whose Index is the Degree of the Prismaticks (by the Construction of Prismaticks.) Hence the Difference in these last Collaterals must be the Product of the Difference in the former (*viz.* of the precedent Triangle) by the same Multiplier.

2°. For the 2d Part, since by the 13th *Theorem*, the Prismatick in the n Place of the 1st Species and 1st Order of the m Degree is $\frac{n^{m+2}+n^{m+1}}{2}$, and that in the n Place of the 2d Species is n^{m+2}, substracting the former from this, the Difference is $n^{m+2} - \frac{n^{m+2}+n^{m+1}}{2} = \frac{2n^{m+2}-n^{m+2}-n^{m+1}}{2} = \frac{n^{m+2}-n^{m+1}}{2}$.

COROLL. The Prismatick in any Place of the 1st Species, and 1st Order, of any Degree, is an Arithmetical Mean betwixt these in the same Place of the 2d Species and 1st Order, of the same and the preceding Degrees; for as that in the given Degree is n^{m+2}, so that in the preceding, or $m-1$ Degree, is $n^{m-1+2}=n^{m+1}$, and the half Difference of these is the Difference betwixt any one of them and their Arithmetical Mean.

THEOREM XVI.

Take any Term of the Triangulars of any Order [numbered from the simple Triangles 1 . 3 . 6 . &c.] and the Collateral Terms of the same Order, of the 1st, or Triangular Species, of as many Degrees as you please, from the 1st successively of Prismaticks; place these orderly in a Series; and under them set the Series of the Coefficients belonging to that Power of a Binomial Root, whose Index is the last Degree taken of the Prismaticks; multiply the corresponding Terms of these two Series together; the Sum of the Products is the common Difference in the Column of Collateral Prismaticks of the next higher Place of the same Order, and last Degree taken in the Prismaticks.

Exam. The Triangular in the 3d Place and 2d Order is 10 (*viz.* $1+3+6$) Also the Prismaticks in the 3d Place, 2d Order, 1st Species, and of the 1st, 2d, and 3d Degrees, are 25, 67, 187. Again, the Coefficients of the 3d Power, are 1 . 3 . 3 . 1; which multiplied into the former produce 473, the common Difference in the Collateral Prismaticks of the 4th Place, 2d Order and 3d Degree; as you'll prove by carrying on the Tables of Prismaticks to the 3d Degree and 2d Order.

$$\begin{array}{c} 10 \;.\; 25 \;.\; 67 \;.\; 187 \\ 1 \;.\; 3 \;.\; 3 \;.\; 1 \\ \hline 10+75+201+187=473 \end{array}$$

DEMON.

Demon. 1°. The common Difference in any Column, as that in the n Place of Collateral Prismaticks, 1st Degree and 1st Order, is equal to the Product of the preceding Triangle multiplied by (n) the Side of the Collateral Prismaticks (by *Theor.* XV.) But that precedent Triangle multiplied by $n-1$ produces the precedent, or $n-1$ Prismatick, 1st Species, 1st Order, and 1st Degree; also if that precedent Triangle, a, is added to it's Product by $n-1$ (*i. e.* to the precedent Prismatick 1st Species) the Sum is the Product of that precedent Triangle by n; or, it is the Difference in these Collateral Prismaticks.

2°. Let the Triangle in the n Place be called a, and the several Prismaticks in the n Place of the 1st Species, 1st Order, of the several Degrees from the 1st, be b, c, d, *&c.* then by the Construction it will be $b=na$. $c=nb$. $d=nc$, &c. Now the Difference in the $n+1$ Column of Collateral Prismaticks, 1st Order, 1st Degree, being $a+b$ (by the 1st Article) the Difference in the $n+1$ Column of the 2d Degree, is the Product of the last Difference $a+b$, by the Side $n+1$ [because the Terms of this Column are Products of $n+1$ by the Terms of that Column in the 1st Degree, whose common Difference is $a+b$] *i. e.* it is $\overline{a+b}\times n+\overline{a+b}=na+nb+a+b$; but $na=b$, and $nb=c$; therefore it is $=a+2b+c$. Again, the Difference in the $n+1$ Place of the 3d Degree is the Product of the last Difference by $n+1$, which is $\overline{a+2b+c}\times n+\overline{a+2b+c}=an+2nb+nc+a+2b+c=a+3b+3c+d$, and so on, as you see ordered in the annex'd Scheme; the Manner of continuing which, shews plainly the Truth of the Theorem; for the Numbers multiplying a, b, c, d, &c. in the several Differences, are evidently the Coefficients of the Powers, whose Indexes express the Degrees of the Prismaticks; so in the 1st, a, b, are multiplied by 1, 1, the Coefficients of the Root or 1st Power of a Binomial. In the 2d Degree, a, b, c, are multiplied by 1, 2, 1, the Coefficients of the 2d Power. In the 3d Degree, a, b, c, d are multiplied by 1, 3, 3, 1, the Coefficients of the 3d Power; and so it's plain they must go on for ever, by the Order of Construction.

a	a	a	a
$b+na$	$2b+na$	$3b+na$	$4b$
. nb	$c+2nb$	$3c+3nb$	$6c$ &c.
.	. nc	$d+3nc$	$4d$
.	.	. nd	e
1st	2d	3d	4th

3°. That the same Thing must be true in the 2d and all the following Orders of any Degree of Prismaticks, is evident from the Construction of these Orders, *viz.* from their being the Sums of the preceding; with this Consideration, that the Difference in the Collaterals of any Order is the Sum of the Differences in all the Collateral Columns of the preceding Order from the corresponding one backwards.

Schol. Besides the Method of finding the Difference in any Column of Collaterals, contained in this Theorem, there is another Method deducible from *Theorem* XIV, which is this: Instead of the Series of Prismaticks of any Degree, 1st Species and 1st Order, take the Series of Differences in the several Columns of the 1st Order of any Degree, and multiply them by the Series of Triangulars mentioned in that Theorem, and in the Manner there explained; the Sum of the Products is the Difference sought: The Reason of which is the same as that for finding the Prismatick of that Degree, Order and Place; because the Difference in any Column is the Sum of all the Differences of the Columns of the preceding Order from the corresponding Place backwards, and therefore have the same Connection and Dependence as the Prismatick Numbers themselves; so that the same Demonstration may be apply'd to this Case, only the first Difference being o, we are to keep out the 1, which is the 1st Term in that Demonstration,

 and

and make a, b, c, &c. represent the several Differences in the preceding Order; and then the Demonstration will be in all respects the same.

COROLLARIES.

1. The Difference in any Column of Collateral Prisms (or Prismaticks, 1st Degree, 1st Order) is the Product of its Side into the preceding Triangle; for that Side being n, and the preceding Triangle a, the preceding Prism is $\overline{n-1}\times a$, and the Difference in the Collateral Prisms of the n Place, is, by this Theorem, $a+\overline{n-1}\times a=a+na-a=na$. Whence again,

2. If to the Triangular Prism in any Place be added the Product of these 3 Numbers, *viz.* the preceding Triangle, the Side of the given Prism, and the Distance of any of its Collaterals, the Sum is the Collateral Prism at that Distance.

THEOREM XVII.

If to any Triangular Prism be added double of its Collateral Triangle, the Sum is equal to 3 times the Collateral Triangular Pyramid.

Exam. The 4th Triangular Prism is 40; the 4th Triangle is 10, whose Double is 20; then $40+20=60=3\times 20$, and 20 is the 4th Triangular Pyramid. Take other Examples out of the annex'd Table.

Triangles	1 .	3 .	6 .	10 .	15
Pyramids	1 .	4 .	10 .	20 .	35
Prisms	1 .	6 .	18 .	40 .	75

DEMON. The Triangle in the n Place is $\frac{nn+n}{2}$ (for it is the Sum of n Terms of the natural Progression,) and the correspondent Prism is $\frac{nnn+nn}{2}$, to which add double the Triangle, *viz.* $nn+n$, the Sum is $\frac{nnn+nn}{2}+n^2+n.=\frac{nnn+nn+2nn+2n}{2}=\frac{nnn+3nn+2n}{2}$. Again, the Triangular Pyramid in the n Place is, by Problem 1st, $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{nnn+3nn+2n}{6}$, Triple of which is equal to $\frac{nnn+3nn+2n}{2}$.

COROLL. Hence we have a particular Rule for finding the Sum of the Series of Triangles, *i. e.* any Term of the Series of Triangular Pyramids, which is this; Take the third Part of the Sum of the corresponding Triangular Prism, and double the Triangle, it is the Sum sought.

THEOREM XVIII.

If to any Triangular Prism be added its corresponding Triangle, the Sum is equal to the Sum of the Collateral Triangular Pyramid, and Square Pyramid.

Exam. 40 is the 4th Triangular Prism, to which add 10, the 4th Triangle; the Sum is $50=30+40$, the 4th Square and Triangular Pyramid.

Triangles	1 .	3 .	6 .	10	
Squares	1 .	4 .	9 .	16	
Tr. Pyram.	1 .	4 .	10 .	20	&c.
Sq. Pyram.	1 .	5 .	14 .	30	
Tr. Prisms	1 .	6 .	18 .	40	

DEMON. The n Triangular Prism is $\frac{n^3+n^2}{2}$, to which add $\frac{n^2+n}{2}$, the n Triangle; the Sum is $\frac{n^3+2n^2+n}{2}$. Again, the n Triangular Pyramid is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$, and the n Square Pyramid is

$2n^3$

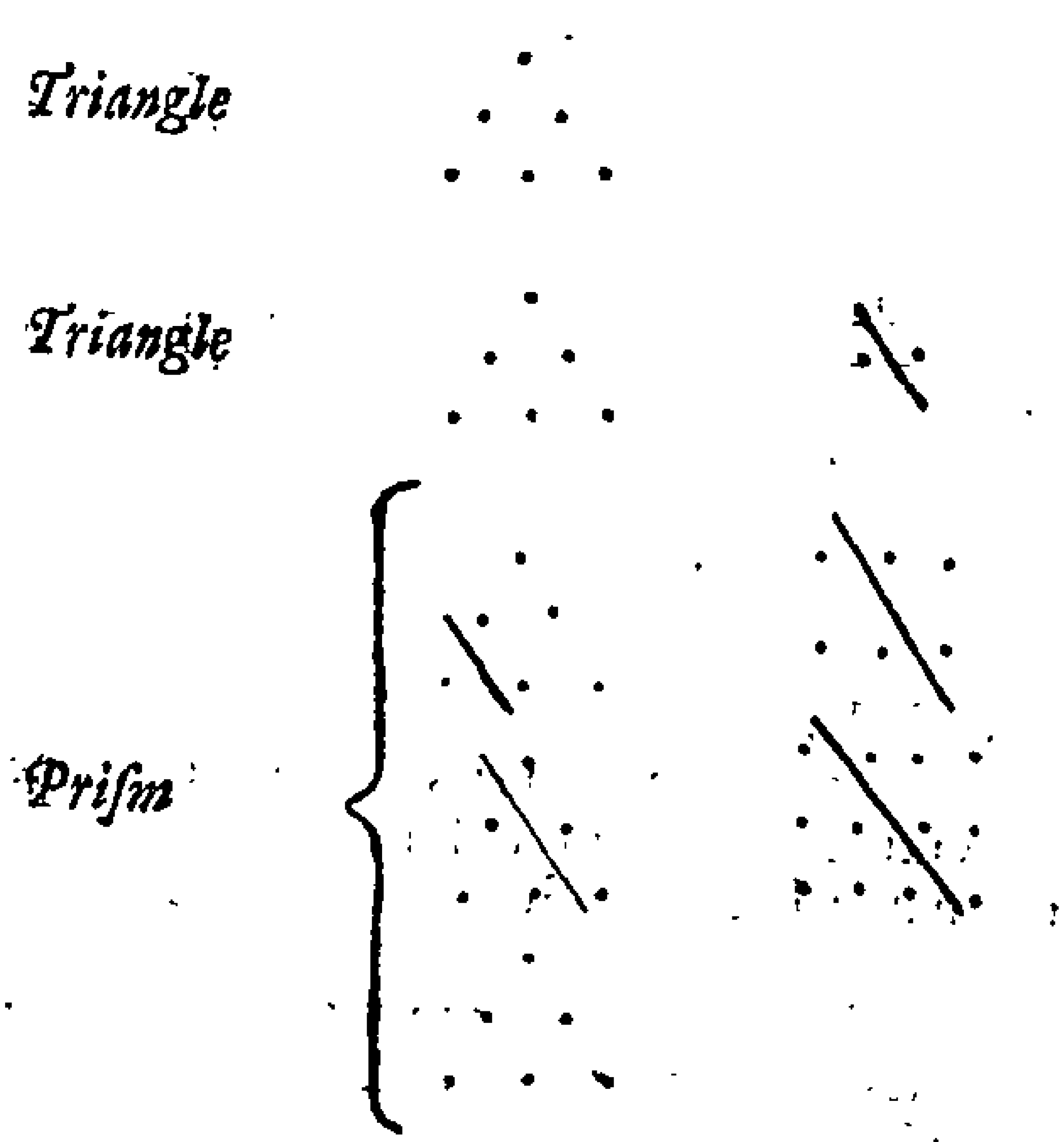
$\frac{2n^3+3n^2+n}{6}$ (as you see in *Schol. Probl.* II.) the Sum of these two is $\frac{3n^3+6n^2+3n}{6}$ $=\frac{n^3+2n^2+n}{2}$, by dividing Numerator and Denominator both by 3.

COROLL. Hence may be deduced a particular Rule for finding the Sum of the Series of Squares 1, 4, 9, &c. *i. e.* for finding any Square Pyramid (or Polygonal of the 2d Species and 2d Order from the Squares.) *Thus*, find the corresponding Triangular Pyramid (by the particular Rule explained in the *Coroll.* of last *Theorem*) and substract this from the Sum of the corresponding Triangle and its Prism; the Difference is the Number sought.

SCHOLIUM.

As the particular Rules contained in the Corollaries to this and the preceding *Theorem*, for finding the Sum of Triangles and Squares, depend immediately upon these Theorems, so they suppose the Truth of some other Rules for finding the same Numbers; which Rules are used in the Demonstration of these Theorems; and therefore to have these Rules demonstrated independently of other Rules for the same Problems, these Theorems must be demonstrated another way; but what I design here is only to observe, That among the antient Writers there is no such Thing as a general Rule for all the Orders and Species of Polygonal Numbers; they have only these two particular Rules for Triangles and Squares, and these they deduce from the same two Theorems, which they demonstrate from the Contemplation of the Schemes or Figures into which the Numbers are disposed; as they have been already explained in the Definitions, and which I shall here represent in the Manner they are formed, to make out these Demonstrations.

For Theorem XVII.

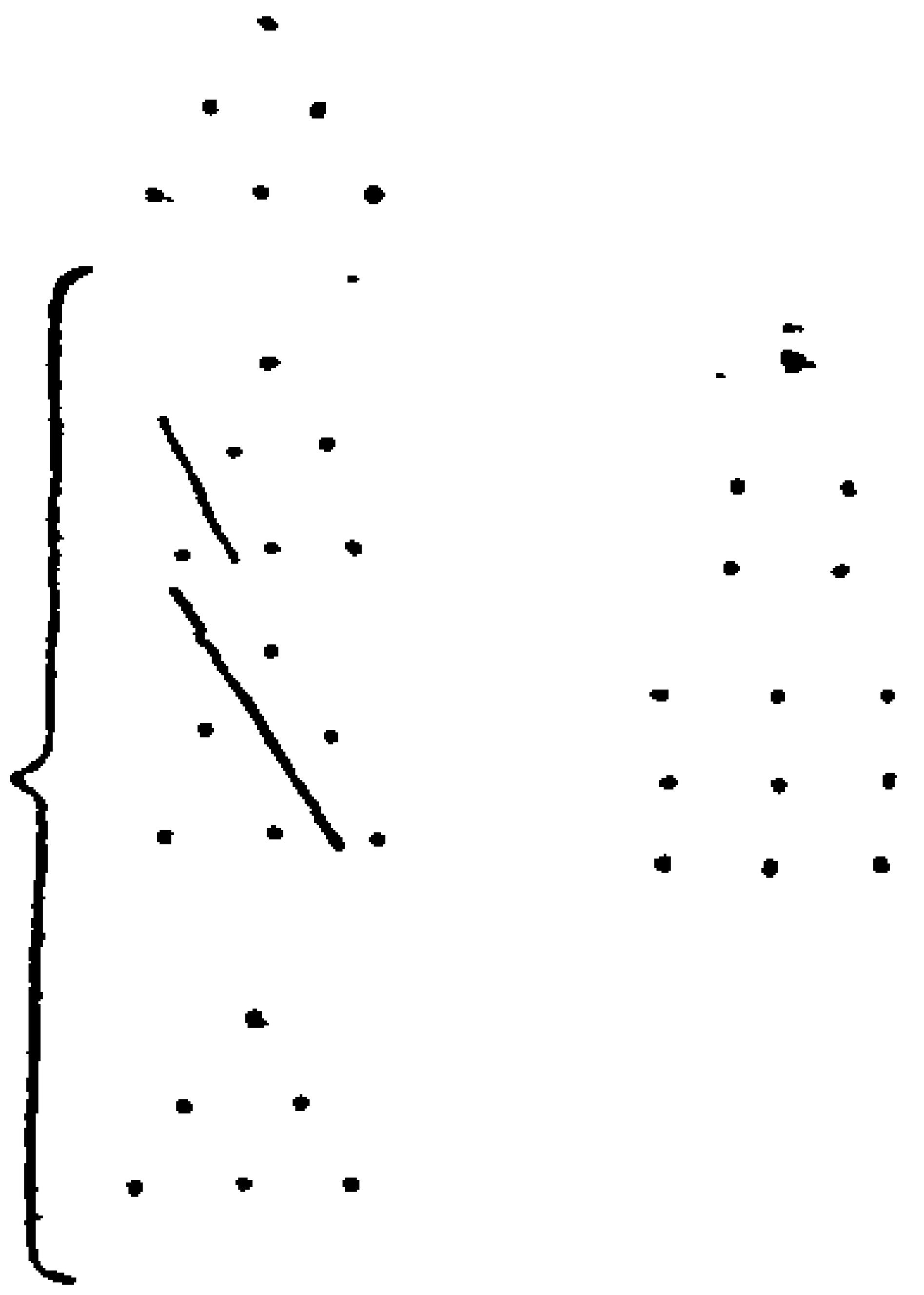


In the 1st Part of this Scheme you have the Triangle 6 taken twice, with its Prism 18; the lower Triangle in its Prism with the Parts cut off from the upper, lying towards the left Hand, do plainly make the Collateral Triangular Pyramids; and the remaining Units on the right, with the two upper Triangles added, make two Pyramids, each equal to the former; as the 2d Part of the Scheme shews, in which the same Lines of Points are only disposed in another Form.

For Theorem XVIII.

Triangle

Prism



In the 1st Part of this Scheme you have the Triangle 6, and its Prism 18; the lower Triangle of which, with the Points cut from the left Hand of the two upper, make the Collateral Pyramid; and the remaining Points, with the Triangle added, make the Collateral Square Pyramid, as the 2d Part of the Scheme shews.

THEOREM XIX.

The Series of Triangular Prisms is the same as the Series of Pentangular Pyramids, as in the annex'd Scheme.

Series	$\div l$	1 . 2 . 3 . 4		*Series*	$\div l$	1 . 4 . 7 . 10
Triangles		1 . 3 . 6 . 10		*Pentagons*		1 . 5 . 12 . 22
Prisms		1 . 6 . 18 . 40		*Pyramids*		1 . 6 . 18 . 40

DEMON. That those two Series will continue to be the same for ever, I thus shew: The Triangular Prism in the n Place is $\frac{n^3+n^2}{2}$. Again, the Pentagonal Pyramid in the n Place, is the Sum of the n Triangular Pyramid, and the Product of the $n-1$ Triangular Pyramid multiplied by 2, the Distance of the Pentagon from the Triangle (*Cor.* 1. *Theor.* II.) But the n Triangular Pyramid is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$ and the $n-1$ Triangular Pyramid is $1\times\frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}=\frac{n^3-n}{6}$, which multiplied by 2 produces $\frac{2n^3-2n}{6}$, to which add the former, *viz.* $\frac{n^3+3n^2+2n}{6}$, the Sum is $\frac{3n^3+3n^2}{6}$ $=\frac{n^3+n^2}{2}$.

THEOREM XX.

Take the Series of Triangles; then the Series of their Squares; and beginning with 1, take the Series of the Differences betwixt each Term and the next of these Squares; they are the Series of Cubes of the natural Progression, as you see in the annex'd Scheme.

Natural Progession	1 . 2 . 3 . 4
Triangles	1 . 3 . 6 . 10 &c.
Their Squares	1 . 9 . 36 . 100
Their Differences	1 . 8 . 27 . 64
which are the Cubes of the 1st Series.	

DEMON. The Sums of the 1st Series taken to $n-1$, and n Terms, are $\frac{n^2-n}{2}$ and $\frac{n^2+n}{2}$; and the Squares of these are $\frac{n^4-2n^3+n^2}{4}$, and $\frac{n^4+2n^3+n^2}{4}$; whose

whose Difference is plainly $\frac{4n^3}{4}=n^3$. Now, since $n-1$, n, may be any two adjacent Terms of the natural Progression, the universal Truth of what is proposed is demonstrated.

COROLLARIES.

1. As every Cube Number is the Difference of some two Squares, (*viz.* whose Roots are the Sum of a Number of Terms of the natural Series, equal to the Root of the given Cube, and the next lesser Sum) so were it required to find two Squares, whose Difference shall be some Cube Number, which is neither known nor assumed, we have here a plain Rule for it; and observe, that I suppose the Difference neither known nor assumed, to make this Problem different from another, which you'll find afterwards, wherein two Squares are found, whose Difference is any given Number.

2. The Sum of the Cubes of the natural Series 1 . 2 . 3 . *&c.* to any Number of Terms, is equal to the Square of the Sum of the same Series taken to the same Number of Terms. Or thus, take the Cubes of the natural Series continually from the beginning, the Sums are all square Numbers, whose Roots are the Sums of the same natural Series taken so far.

Schol. As the last Corollary follows plainly from the Theorem, so if that Corollary is demonstrated another way, the Theorem will as clearly follow from it; and as to the Demonstration of the Corollary independently of this Theorem, I have found one, which, though not so simple as the preceding Demonstration of the Theorem, yet is curious enough, and therefore worth the explaining here.

The Thing then to be demonstrated is this, *viz.* that $1^3+2^3+3^3+4^3$, *&c.* $=\overline{1+2+3+4}^2$, *&c.*

1*st*. From the Nature of Multiplication it's plain, that the Square of any Multinomial Root (*i. e.* a Root consisting of many Parts) is equal to the Sum of the Squares of each Part of the Root, and twice the Product of every Pair of Members of the Root; as in the annex'd Scheme; wherein $a+b+c+d$, &c. being squared, the Square is the Sum of all the Squares and Products set under it; and by the Order in which they are disposed, you see the Square is composed of as many Members as the Root, each of which is equal to the Square of one Member of the Root, and the Sum of the double Products of that into all the preceding Members (on the left Hand) which evidently comprehends the double Products of every Pair of Members of the Root.

$$\begin{array}{rrrr}
a+ & b+ & c+ & d\text{, \&c.}\\
a+ & b+ & c+ & d\text{, \&c.}\\
\hline
a^2+ & b^2+ & c^2+ & d^2\\
 & 2ab+ & 2bc+ & 2cd\\
 & & 2ac+ & 2bd\\
 & & & 2ad
\end{array}$$

2*d*. If the Parts of the Multinomial Root are the several Terms of the natural Series 1 . 2 . 3 . *&c.* then suppose the Square of it is taken in the Manner of the preceding Scheme, I say that each Member of it (*i e.* the Sum of the Square of each Term of the Root, and the double Products of that Term into all the preceding Terms) is equal to the Cube of that Term. For *Exam.* The Cube of 4 is 64, equal to the Square of 4 (*viz.* 16) more $2\times3\times4+2\times2\times4+2\times1\times4$ $(=24+16+8=48.)$ And that it is so universally I thus prove, (1°.) It's plain, that to multiply any Term of the Root into each preceding Term severally, the Sum of the Products is equal to the Product of that Term into the Sum of all the preceding Terms; and because we must double all these Products, therefore we must take double the Sum of the preceding Terms, and multiply it into the given Term. Then, (2°.) This Product is equal to the Square of the given Term multiplied into

$$\begin{array}{rrrr}
1+ & 2+ & 3+ & 4\text{, \&c.}\\
1+ & 2+ & 3+ & 4\text{, \&c.}\\
\hline
1+ & 4+ & 9+ & 16\\
 & 4+ & 12+ & 24\\
 & & 6+ & 16\\
 & & & 8\\
\hline
1+ & 8+ & 27+ & 64
\end{array}$$

into the next preceding Term, by the Rules of Progressions; for *Example*, $6\times2\times\overline{1+2+3+4+5}=6\times6\times5$; because $\overline{1+2+3+4+5}=\frac{6\times5}{2}$; and this multiplied by 2, produces 6×5. *Universally*, Call the Term next preceding any given one, l, that given one must be $1+l$; which is also the Sum of the Extremes of that Series, whereof l is the greatest Extreme, *i. e.* of the Series to be summed; but this Sum is $\overline{1+l}\times\frac{l}{2}$ (l being here the Number of Terms) and the Double of this is $\overline{1+l}\times l$, which multiplied into the given Term, $1+l$, produces $\overline{1+l}^2\times l$. But, (3°.) It's manifest that $\overline{1+l}^2\times l$ (the Sum of twice the Product of $1+l$, the given Term, into all the preceding Terms) added to $\overline{1+l}^2$ (the Square of the given Term) makes $\overline{1+l}^2\times l+\overline{1+l}^2=\overline{1+l}^2\times\overline{1+l}=\overline{1+l}^3$, the Thing to be shewn.

3*d*. Since the Square of the Multinomial, $1+2+3+4$, &c. is resolveable into as many Members as the Root, each of which is proved to be equal to the Cube of a different Member of the Root; therefore that Square is equal to the Sum of the Cubes of the several Members of the Root, *i. e.* $1^3+2^3+3^3$ &c. $=\overline{1+2+3}^2$ &c.

Theorem XXI.

Take the Series of square Pyramids (*i. e.* the Sums of the natural Series of Squares) multiply each of them by 3; and from the Series of Products, take the Series of the Sums of the 2 Series of Triangles and Squares, as in the Margin; the Series of the Differences is the Series of Cubes of the natural Progression.

1	2	3	4	5	*Natural Series*
1	3	6	10	15	*Triangles*
1	4	9	16	25	*Squares*
1	5	14	30	55	*Square Pyramids*
3	15	42	90	165	*Their Product by* 3
2	7	15	26	40	*Sum of Trian. and Squ.*
1	8	27	64	125	*Cubes*

Demon. The n Square Pyramid is $\frac{2n^3+3n^2+n}{6}$ (by *Probl.* II. *Schol.*) which multiplied by 3 produces $\frac{2n^3+3n^2+n}{2}$. The n Triangle is $\frac{nn+n}{2}$, and n Square is nn: Then $n^2+\frac{nn+n}{2}=\frac{3n^2+n}{2}$; which taken from the former, leaves plainly $\frac{2n^3}{2}=n^3$. And because n may be any Term, therefore the *Theorem* is true.

Theorem XXII.

Take the Series of Heptangular Pyramids, and to them add the Series of Triangular Pyramids in this Manner, *viz.* The 1st of the Triangulars to the 3d of the other; the 2d Triangular to the 4th of the other; and so on; *i. e.* universally, the n Heptangular Pyramid added to the $n-2$ Triangular. All these Sums are Cubes; and prefixing the first two Heptangular Pyramids, 1 . 8 (which are Cubes) you have the whole Series of Cubes.

Series ∸ *l*	1 . 2 . 3 . 4 . 5
Triangles	1 . 3 . 6 . 10 . 15
Series ∸ *l*	1 . 6 . 11 . 16 . 21
Heptag.	1 . 7 . 18 . 34 . 55
Hept. Pyramids	1 . 8 . 26 . 60 . 115
Triang. Pyramids	1 . 4 . 10
Cubes	1 . 8 . 27 . 64 . 125

Demon. The n Triangular Pyramid is $\frac{n^3+3n^2+2n}{6}$, and the $\overline{n-1}$ Term is $\frac{n^3-n}{6}$ (*Probl.* II. *Schol.*) But the Distance of the Heptagon from the Triangle, is 4; therefore (by *Probl.* II.) the n Heptangular Pyramid is $\frac{n^3+3n^2+2n}{6}+\frac{n^3-n}{6}\times 4=\frac{n^3+3n^2+2n+4n^3-4n}{6}=\frac{5n^3+3n^2-2n}{6}$. Again the $n-2$ Triangular Pyramaid is $1\times\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n}{3}=\frac{n^3-3n^2+2n}{6}$; to which add the n Heptangular Pyramid, *viz.* $\frac{5n^3+3n^2-2n}{6}$; the Sum is $\frac{6n^3}{6}=n^3$.

Theorem XXIII.

Take the Series of Octangular Pyramids, and from them substract the Series of Triangles; thus, The 1st Triangle from the 2d Octangular Pyramid, and so on, *i. e.* universally, the $n-1$ Triangle from the n Pyramid; the Series of Differences is the Series of Cubes after 1.

Series ∸ *l*	1 . 7 . 13 . 19
Octogons	1 . 8 . 21 . 40
Octog. Pyramids	1 . 9 . 30 . 70
Triangles	1 . 3 . 6
Cubes	1 . 8 . 27 . 64

Demon. The n Triangular Pyramid is $\frac{n^3+3n^2+2n}{6}$, and the $\overline{n-1}$ Triangular Pyramid is $\frac{n^3-n}{6}$: Therefore the n Octangular Pyramid is (by *Probl.* II.) $\frac{n^3+3n^2+2n}{6}+\frac{n^3-n}{6}\times 5=\frac{n^3+3n^2+2n+5n^3-5n}{6}=\frac{6n^3+3n^2-3n}{6}$; from which, substract the $n-1$ Triangle, which is $\frac{n^2-n}{2}$; the Difference is $\frac{6n^3+3n^2-3n}{6}-\frac{3n^2-3n}{6}=\frac{6n^3}{6}=n^3$.

	0 . 6 . 12 . 18
	0 . 6 . 18 . 36
	0 . 6 . 24 . 60
	1 . 2 . 3 . 4
	1 . 8 . 27 . 64

Coroll. From this it is plain, That if we take a Series, ∸ *l*, beginning with *o*, and proceeding by the Difference 6; then take the Sums of this Series; and then the Sums of these Sums; and to this last Series of Sums add the natural Series, 1 . 2 . 3 . &c. the last Sums make the Series of Cubes. The Deduction of the universal Truth of which from the present *Theorem*, is easily made; thus, In the Scheme of the *Theorem*, each Term of the 1st Series is 1 more than its Collateral in the 1st Series of the present Scheme; therefore each Term of the 2d Series of the former Schemes, exceeds the Collateral of the 2d Series of this Scheme, by as many Units as the preceding Number of Terms. Hence again; Each Term of the 3d Series, former Scheme, exceeds the Collateral in the 3d Series of this Scheme, by the Sum of as many Terms of the natural Series, as the Number of preceding Terms: Whence all the rest is plain.

From

0	6	12	18
1	7	19	37
1	8	27	64

From this laſt, *again*, it will eaſily appear, That if we take the Series, 0 . 6 . 12 . &c. and taking the Sum of it, add 1 to each Sum; and then of this Series of Sums take the Series of Sums; theſe are the Cubes of the natural Progreſſion; for by taking the Sums of the Series, 0 : 6 : 18 : 36 : &c. after 1 is added to each (*i. e.* the Sums of 1 . 7 . 19 . &c.) it's plain the Sums at every Step, will be more than the Sums taken without theſe Units (*i. e.* than the Sums of the Series, 0 . 6 . 18 . &c.) by as many Units as the Number of Terms added: So that it will be the ſame as if taking the Sums of this Series, 0 . 6 . 18 . 36, we add to the Series of the Sums the natural Series, 1 . 2 . 3; as in the preceding Scheme.

Theorem XXIV.

Take the Series of odd Numbers, 1 . 3 . 5 . &c. and out of this make another Series; thus, Take the 1ſt Term; then the Sum of the next 2 Terms; again, the Sum of the next 3 Terms, after the laſt; and ſo on, taking in at every Step, one Term more; ſo that the Number of Terms taken at every Step, are the natural Series, 1 . 2 . 3 . &c. The Series thus compoſed, is the Series of Cubes of the natural Series; each Sum being the Cube of the Number of Terms added together.

$$1 \,.\, \underbrace{3 \,.\, 5}_{} \,.\, \underbrace{7 \,.\, 9 \,.\, 11}_{} \,.\, \underbrace{13 \,.\, 15 \,.\, 17 \,.\, 19}_{} \,.\, \underbrace{21 \,.\, 23 \,.\, 25 \,.\, 27 \,.\, 29}_{}$$

Cubes	1 .	5 .	27 .	64 .	125	&c.
Roots	1 .	2 .	3 .	4 .	5	

Demon By *Theor.* XX. *Cor.* 2. the Sum of the Series of Cubes, is the Square of the Sum of the natural Series taken to the ſame Number of Terms: But the Sums of the odd Series are the Squares of the ſeveral Number of Terms ſummed (*Theor.* IX.) And in this Scheme there are always as many odd Terms from the Beginning, as the Sum of all the Roots; therefore the Sum of all theſe odd Numbers being the Square of the Sum of all the Cube Roots, is the Sum of all the Cubes. Again, That the Sum of the ſeveral Terms, diſtinguiſhed in the Series of odd Numbers, are the particular Cubes of the Number of theſe Terms, is evident; becauſe 1 is the Cube of 1, and 1+3+5, the Sum of the Cubes of 1 and 2, by what's firſt ſhewn; therefore 3+5 is the Cube of 2. Then ſince 1+3+5+7+9+11 is the Sum of the Cubes of 1, 2, 3, therefore 7+9+11 is the Cube of 3. And ſo on.

Another Demon. This *Theorem* may alſo be demonſtrated independently of *Theor.* XX. by the Conſideration of the odd Series. Thus,

1°. If the Root of any Cube is an odd Number, its Square is alſo an odd Number; and is therefore a Term of the odd Series: Then taking as many adjacent Terms of the odd Series, as the Root of any Cube expreſſes, and whereof the Square of the ſame Root is the middle Term; it's plain the Sum of all theſe Terms will be the Cube; becauſe, being an Arithmetical Progreſſion, the Sum of every two Terms, equally diſtant from the middle one, is double that middle one, and conſequently the whole Sum is equal to as many Times the middle Term as the Number of Terms expreſſes; but that middle Term is the Square; and the Number of Terms the Root; therefore the Whole is the Product of the Square by the Root, *i. e.* the Cube.

2°. If the Root is an even Number, then we can find in the odd Series two adjacent Terms; the one exceeding the Square by 1, and the other wanting 1 of it; conſequently, taking as many Terms as the Root expreſſes, whereof theſe two now mentioned are the two middle Terms; the Sum of the Whole is the Cube; for the Sum

Sum of the two middle, and of every other two equally distant from them, are each double the Square; and so the Whole is equal to the Product of the Square by the Root, *i. e.* is the Cube.

3°. Representing the two preceding Cases in the annex'd Scheme [wherein a being any odd Number or Root, the 1st Series represents a Number of adjacent odd Numbers, whose Sum is a^3; and a being an even Number, the 2d Series is a Number whose Sum is a^3; understanding these Series to contain as many Terms as the Root has Units, whereof a^2 is the middle one in the 1st, and a^2-1, a^2+1, the two middle ones in the 2d Case.]. Hence it will be evident, that the least and greatest Terms in the particular Series of adjacent odd Numbers, whose Sum is the Cube of the Number of Terms, are thus expressed, *viz.* The least a^2+1-a, and the greatest a^2+a-1: For the Root being odd, the Number of Terms is odd; and there are as many Terms on each Hand of the Middle, a^2, as Half the Root -1 or $\frac{a-1}{2}$; and the common Difference being 2, the Number substracted from, or added to a^2 in the Extremes, is equal to 2, taken as oft as $\frac{a-1}{2}$ expresses: But $\frac{a-1}{2}\times 2=a-1$; therefore the Extremes are $a^2-\overline{a-1}$, and $a^2+\overline{a-1}$, which are a^2-a+1, and a^2+a-1. Again, If a is an even Number, the two middle Terms being always a^2-1, a^2+1, and the Number of Terms below a^2-1, and above a^2+1, being half of $a-2$, or $\frac{a-2}{2}$; it follows, that the Extremes are a^2-1, wanting the Product of 2 by $\frac{a-2}{2}$, which Product is $a-2$; and a^2+1, with the same Number added to it; which Extremes are therefore $a^2-1-\overline{a-2}=a^2-1-a+2=a^2-a+1$, and $a^2+1+a-2=a^2+a-1$.

$$\text{\&c. } a^2-6 : a^2-4 : a^2-2 : a^2 : a^2+2 : a^2+4 : a^2+6 : \text{\&c.}$$
$$\text{\&c. } a^2-5 : a^2-3 : a^2-1 : a^2+1 : a^2+3 : a^2+5 : \text{\&c.}$$

4°. Take any two Roots differing by 1, as a, and $a+1$; the greatest Extreme of the Series of odd Numbers, whose Sum is the Cube of a, is, by the last, a^2+a-1. And the least Extreme of the Series, whose Sum is the Cube of $a+1$, is (by substituting $a+1$ instead of a, in this Expression, a^2-a+1) $=\overline{a+1}^2-\overline{a+1}+1=a^2+2a+1-a$ $=a^2+a+1$. Compare this with the former, *viz.* a^2+a-1, it's plain their Difference is 2, *i. e.* the greatest Term of these adjacent odd Numbers whose Sum is a^3, and the least of these whose Sum is $\overline{a+1}^3$, differ by 2; and so they are two adjacent odd Numbers; consequently the Series for the one Cube begins at the Term next after that one with which the other ends; and therefore the Series of Cubes are found in the Manner prescribed in the *Theorem.*

Schol. As this *Theorem* is demonstrated by Means of *Theor.* XX. so that is demonstrable by Means of this, and *Theor.* IX.

Theorem XXV.

Take the Series of Pentagonal Pyramids, and multiply each of them by 3; then take the Series of Squares, and multiply each of them by 2: Substract the last Series of Products from the former; the Differences are the Series of Pentagonal Prisms.

Series $\div l$	1 .	4 .	7 .	10
Pentagons	1 .	5 .	12 .	22
Pent. Pyramids	1 .	6 .	18 .	40
Products by 3	3 .	18 .	54 .	120
Squares	1 .	4 .	9 .	16
Products by 2	2 .	8 .	18 .	32
Pent Prisms	1 .	10 .	36 .	88

DEMON. Pentagons proceed from the Series, 1 . 4 . 7 . &c. whose common Difference is 3; of which therefore the nth Term is $1+\overline{n-1}\times 3 = 3n-2$; and the Sum of the Extremes is $\overline{3n-1}$. *Lastly*, The Sum of n Terms is $\overline{3n-1}\times\frac{n}{2}=\frac{3n^2-n}{2}$, which is the n Pentagon; and this multiplied by n, produces $\frac{3n^3-n^2}{2}$, the n Pentagonal Prism. Again, The nth Triangular Pyramid is $\frac{n^3+3n^2+2n}{6}$, and the $n-1$ Triangular Pyramid is $\frac{n^3-n}{6}$; the Distance of the Triangle and Pentagon is 2, and $\frac{n^3-n}{6}\times 2=\frac{2n^3-2n}{6}$; wherefore (by *Cor. Theor.* II.) the n Pentagonal Pyramid is $\frac{n^3+3n^2+2n}{6}+\frac{2n^3-2n}{6}=\frac{3n^3+3n^2}{6}=\frac{n^3+n^2}{2}$; which multiplied by 3, makes $\frac{3n^3+3n^2}{2}$; from which take $2n^2$, the Remainder is $\frac{3n^3-n^2}{2}$, the n Pentagonal Prism.

General SCHOLIUM.

Though there be no general *Canon*, that I know, for finding the Sum of any Series of Prismaticks of any Degree, Species and Order, excepting what is contained in the XIVth *Theorem*, which supposes the Series of Prismaticks of the 1st Order of the same Species and Degree, and to the same Number of Terms, as that whose Sum is sought; yet for the Sums of the Prismaticks of the 2d Species, 1st Order, of any Degree, *i. e.* for the Sums of Powers of the natural Progression, from the Cubes and upwards, we can, by Means of the preceding *Theory* of Polygonals, investigate particular *Canons* for every different Power; whereby the Sum may be found, having only the Number of Terms.

In order to which *observe*, That as the Invention of these *Canons*, for any Degree, depends upon the *Canons* for the preceding Degrees; so, though Squares are Polygons, and not Prismaticks, yet it will be necessary to take the Squares within the following *Problem*, that we may more easily, by the two first and most simple Powers, understand the Method of Investigation for all the superior Powers.

Again, *observe*, That we have already explained the particular Canons for the Sums of Squares and Cubes; that for Squares, being $\frac{2n^3+3n^2+n}{6}$, as you see in the 2d Article of the *Scholium* after *Problem* II. and that for Cubes, being $\frac{n^4+2n^3+nn}{4}$; for by *Theorem* XX. *Coroll.* 2. the Sum of the Cubes of the natural Progression, to the nth Term, is the Square of the Sum of the Roots; but this Sum is $\frac{n^2+n}{2}$, whose Square is $\frac{n^4+2n^3+n^2}{4}$. But the Method of investigating Rules for the higher Powers being different from the Method, by which these Rules for Squares and Cubes have already been invented, and depending also upon a new Method, by which the same two Rules may be investigated: Therefore we must explain this other

other Method for Squares and Cubes; and this being done, the universal Method for all other Powers will be easily understood.

Observe in the last Place, That there are several Methods of finding these Canons, so as to make the Invention of the Rule for any Degree, depend upon the Rules for the preceding Degrees; but I confine my self to that which is most natural in this Place, *viz.* which depends upon the universal Rule for the several Orders of Polygonals of the Triangular Kind. You'll find another curious Method in *Ronayne*'s Algebra.

PROBLEM III.

How to *investigate Rules* for finding the Sums of the Series of Powers of any Degree of the natural Progression, 1 . 2 . 3 . &c. having the Number of Terms only given.

SOLUTION.

1*st*. For the Sum of the Squares, $1+4+9+$&c. to the nth Term. The nth Triangular of the 3d Order (from Units) *i. e.* the Sum of the *Triangles*, to the nth Term, is $1\times\frac{n}{1}\times\frac{n+1}{2}=\frac{n^2+n}{2}$ (by the 2d Rule for *Probl.* I. with the *Coroll.* to *Theorem* III. Or, see the *Scholium* after that Rule.) Again, Take n, successively equal to 1 . 2 . 3 . &c. and applying this Canon, we have $\frac{1^2+1}{2}$, $\frac{2^2+2}{2}$, $\frac{3^2+3}{2}$, &c. which express the Series of Triangulars of the 3d Order, to any Number of Places given; which Number being called n, the Sum of this Series is the nth Triangular of the 4th Order; but the n Triangular of the 4th Order is, by the same general Rule, $1\times\frac{n}{1}\times\frac{n+1}{1}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$, which is therefore equal to $\frac{1^2+1}{2}+\frac{2^2+2}{2}+\frac{3^2+3}{2}$ &c. carried to n Terms; and this is $=\frac{1}{2}$ of $\overline{1^2+1+2^2+2+3^2+3}$ &c. $=\frac{1}{2}$ of $\overline{1+2^2+3^2}$ &c. $+\frac{1}{2}$ of $\overline{1+2+3}$ &c. Now $1+2+3$ &c. to n Terms, is $=\frac{nn+n}{2}$; therefore $\frac{1}{2}$ of it is $=\frac{nn+n}{4}$; consequently $\frac{n^3+3n^2+2n}{6}=\frac{1}{2}$ of $\overline{1+2^2+3^2}$ &c. $+\frac{n^2+n}{4}$; and substracting $\frac{nn+n}{4}$ from both Sides, the Remainders are equal, *viz.* $\frac{1}{2}$ of $\overline{1+2^2+3^2}$ &c. $=\frac{n^3+3n^2+2n}{6}-\frac{n^2+n}{4}=\frac{4n^3+12n^2+8n-6n^2-6n}{24}=\frac{4n^3+6n^2+2n}{24}$; and multiplying the first and last Expressions, both by 2, the Products are $1+2^2+3^2$ &c. $=\frac{8n^3+12n^2+4n}{24}=\frac{2n^3+3n^2+n}{6}$; which is the Rule for the Sum of the Squares, *viz.* $1+2^2+3^2$ &c.

2*d*. For the Sum of the Cubes, or $1^3+2^3+3^3+$&c. to the nth Term. The nth Triangular of the 4th Order (from Units) *i. e.* the Sum of the Triangulars of the 3d Order to the nth Term, is $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$; and taking n, successively equal

 to

to 1, 2, 3, &c. the Series of Triangulars of the 4th Order are, $\frac{1^3+3\times1^2+2\times1}{6}$, $\frac{2^3+3\times2^2+2\times2}{6}$, $\frac{3^3+3\times3^2+2\times3}{6}$, &c. to any Number of Terms; which Number being called n, the Sum of this Series is the nth Triangular of the 5th Order; which, by the general Rule, is equal to $1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}=\frac{n^4+6n^3+11n^2+6n}{24}$; but the former Series (equal to this) is the Sum of these 3 Parts, *viz.* $\frac{1}{6}$ of $1+2^3+3^3$ &c. $+\frac{3}{6}$ of $1+2^2+3^2$ &c. $(=\frac{2n^3+3n^2+n}{6}$, by the last$)+\frac{2}{6}$ of $1+2+3$ &c. $\left(=\frac{nn+n}{2}\right)$ and taking the last two Parts from both, we have $\frac{n^4+6n^3+11n^2+6n}{24}-\frac{3}{6}$ or $\frac{1}{2}$ of $\frac{2n^3+3n^2+n}{6}-\frac{2}{6}$ or $\frac{1}{3}$ of $\frac{n^2+n}{2}=\frac{1}{6}$ of $1+2^3+3^3$ &c. And reducing the first Part to the most simple Expression, by the common Rules of Fractions; and then multiplying both Parts by 6, there will come out $1+2^3+3^3$ &c. $=\frac{n^4+2n^3+n^2}{4}$; the Rule for Cubes.

Now for all superior Powers, the Investigation of Rules, by the same Method, will be plain and obvious to such as understand the first two Cases, how they depend upon one another, and upon the General Canon for Triangular Numbers: For we gradually express, by that Canon, the nth Triangular of the Order (from Units) expressed by 1 more than the Index of the Power (as above, we took the 3d Order for Squares, and the 4th Order for Cubes;) and taking n, gradually equal to 1, 2, 3, &c. we express the several Terms of the Triangulars of that Order, according to that Canon, to any Number of Terms, as n; and the Sum of all these Terms will resolve into as many other Series (connected by Addition, and in some Cases, some of them substracted) as there are Members in the Numerator of the General Canon, and whereof one of them will be some Multiple or *aliquot* Part of the Sum of the Series of Powers sought, to n Terms; and the other Parts are some Multiple or *aliquot* Part of the Sums of as many Terms of some of the inferior Powers (of the same natural Progression, 1 . 2 . 3 . &c.) which we must express by the Rules already invented for these inferior Powers; then we consider that the Sum of this Series, thus expressed, is the nth Term of the next Order of Triangulars; and this we express by the same General Rule as before we did the nth Term of the preceding Order: And then comparing these two different Expressions of the same Number, we find, after a due Reduction, the most simple Expression for the Sum of the Powers sought.

I shall here give you the Canon for Biquadrates, and leave the Investigation and Proof of it to your Exercise.

The *Canon* is, $1+2^4+3^4+4^4$ &c. $=\frac{6n^5+15n^4+10n^3-n}{30}$.

§. III. *Of* Oblongs, *and some remaining curious Propositions concerning* Squares.

THEOREM XXVI.

IF we take the Series of even Numbers, 2 . 4 . 6 . &c. and then the Series of their Sums, 2 . 6 . 12 . &c. these are the Series of *Oblongs*.

DEMON.

2	4	6	8	
2	6	12	20	
1	2	3	4	5

DEMON. The 1st even Number is also the 1st Oblong, 2. And that all the other Sums are the other Oblongs, is thus plain. By the Nature of $\div l$ Progressions, the Sum of the even Numbers to n Terms is $nn+n=\overline{n+1}\times n$, which is therefore an Oblong; and taking n, gradually greater by 1, it's plain that $\overline{n+1}\times n$ becomes gradually the next Oblong.

THEOREM XXVII.

Double the several Terms of the Series of Triangles (*i. e.* of the Sums of the natural Series 1 . 2 . 3 . *&c.*) and the Products are the Series of Oblongs.

1	2	3	4	
1	3	6	10	*Triangles*
2	6	12	20	*Oblongs*

DEMON. Take any 2 adjacent Numbers n, $n+1$; in the natural Series; their Product is an Oblong, and it is also double the Sum of the natural Series, to n Terms; for that Sum is $\frac{n\times\overline{n+1}}{2}$; consequently taking n, gradually 1 more, we have the whole Series of Oblongs.

THEOREM XXVIII.

The Arithmetical Means betwixt (or half Sums of) every two adjacent Oblongs, make the Series of natural Squares after 1, the Root of each Square being the lesser Side of the greater Oblong, and the greater Side of the lesser; and is also the common Difference betwixt the Square and each of these Oblongs.

	1	2	3	4	5	6	
Obl.	2	6	12	20	30		*&c.*
	1	4	9	16	25		

DEMON. Take 3 Terms in the natural Series $n-1$. n. $n+1$. the two adjacent Oblongs produced from these, are $\overline{n-1}\times n = n^2-n$, and $n\times\overline{n+1}=n^2+n$, and their Sum is $2n^2$, whose half is n^2, the Arithmetical Mean. Now n being the middle Term of 3, assumed in the natural Series, which in the first 3, (*viz.* 1 . 2 . 3) is 2, whose Square is 4; and every successive Term of the natural Series being the middle Term of the next three, from which the two next adjacent Oblongs proceed; hence the universal Truth of the *Theorem* is clear.

COROLLARIES.

1. The Sum and Difference of any Square and its Root, are the 2 Oblongs next greater and lesser than that Square.

2. Take the Series of Oblongs, and of Squares from 4; the Differences of the corresponding Terms of these Series are the natural Progression of Numbers, from 2; thus, $4-2=2$. $9-6=3$. $16-12=4$. and so on.

THEOREM XXIX.

The Series of Oblongs is the Series of Geometrical Means betwixt every two adjacent Squares, (see the preceding Scheme.) The common Ratio of that Mean, and these Extremes, being that of the Roots of these Squares, which are the Sides of the Oblong.

DEMON. Betwixt any two Squares AA, and BB, the Geometrical Mean is AB (for AA : AB : : AB : BB, the Ratio being A : B) wherefore if the Roots A, B, differ by 1, the Geometrical Mean AB is an Oblong; and thus consequently we have the whole Series of Oblongs.

Coroll. Take the Series of Oblongs and of the Squares; compare the 1st Square, 1, to the 1st Oblong 2; and the 2d to the 2d, and so on. The Ratios are the continued Ratios of the natural Progression, *viz.* 1 : 2. 2 : 3. 3 : 4. &c.

Theorem XXX.

Take the Series of Squares and of Oblongs; out of these make another Series, thus; Take the 1st Square 1, then to this Square add the 1st Oblong, then to the 1st Oblong add the 2d Square, and so on, adding each Square to the next Oblong, and each Oblong to the next Square; I say the Sums are the Series of Triangles, each Sum being a Triangle, which is the Sum of such a Number of Terms of the natural Series, as is equal to the Sum of the Places of the Square and Oblong added, in their respective Series: Thus, 16 is the 4th Square, and 12 the 3d Oblong, and their Sum is $16+12=28$, the 7th Triangle.

Nat. Series	1 . 2 . 3 . 4 . 5
Their Squares	1 . 4 . 9 . 16 . 25
Oblongs	2 . 6 . 12 . 20
Triangles	1 . 3 . 6 . 10 . 15 . 21 . 28 . 36 . 45

Demon. It is plain that the Places of the two Terms added in their several Series, are alternately equal, and then differing by 1, the Place of the Square being the greatest Number; for the 1st Square is added to the 1st Oblong, then the 1st Oblong to the 2d Square, and so on; therefore the Thing to be proved is, universally that the n Square $+$ n Oblong, is the Sum of $2n$ Terms of the natural Series; and that the n Oblong $+$ $\overline{n+1}$ Square, is the Sum of $2n+1$ Terms of the natural Series: Which is shewn thus; the n Oblong is $n\times\overline{n+1}=nn+n$, to which add nn, the Sum is $2nn+n$; and the Sum of the $2n$ Terms of the natural Series is $\overline{2n+1}\times\frac{2n}{2}=\overline{2n+1}\times n=2nn+n$; which is the first Thing. Again, the n Oblong being $nn+n$, and the $\overline{n+1}$ Square being $nn+2n+1$, their Sum is $2nn+3n+1$; and the Sum of $2n+1$ Terms of the natural Series is $\overline{2n+2}\times\frac{2n+1}{2}=\overline{2n+1}\times\overline{n+1}=2n^2+3n+1$; which is the second Thing.

Or the Demonstration may be made thus; each Square is the Sum of as many Terms of the odd Series, as the Root expresses (*Theor.* IX.) and the Oblong in the next lower Place is the Sum of all the intermediate even Numbers; also the Oblong of the same Place is the Sum of all the intermediate even Numbers, and the next greater even Number (*Theor.* XXVI.) whence the *Theorem* is evident. See the following Scheme, wherein $9=1+3+5$, and $6=2+4$, therefore $9+6=1+2+3+4+5$; also $16=1+3+5+7$, and $20=2+4+6+8$, therefore $16+20=1+2+3+4+5+6+7$; and so of others.

Odd Series	1 . 3 . 5 . 7 . 9
Even	2 . 4 . 6 . 8
Squares	1 . 4 . 9 . 16 .
Oblongs	2 . 6 . 12 . 20
Triangles	1 . 3 . 6 . 10 . 15 . 21 . 28 . 36 . 45

Theorem XXXI.

Take the Sum of every two adjacent Oblongs, and add it to the Double of the Arithmetical Mean (or Interjacent Square) these Sums make the Series of Squares of the even Numbers after 2. Thus, $2+6+8=16$, the Square of 4; then $6+12+18=36$, the Square of 6.

Demon.

Squares	1	. 4	. 9	. 16			
Oblongs		2	. 6	. 12	. 20		
Squares			16	. 36	. 64		
Roots			4	. 6	. 8		

Demon. The Sum of every two adjacent Oblongs being double the Arithmetical Mean, or interjacent Square, which call nn; therefore double that Square added to that Sum, is 4 Times that Square or $4nn$; but 4 and nn being both Squares, their Product $4nn$ is also a Square, whose Root is $2n$, an even Number; also the first Value of n being 2, in that Case $2n$ is $=4$; and the following Values of n being gradually 1 more (because the Roots of the Squares added are 1 . 2 . 3, &c.) therefore $2n$ is at every Step 2 more than in the preceding; and so they make the even Series, 4 . 6 . 8, &c.

Theorem XXXII.

Take the Sum of every two adjacent Squares, and twice the interjacent Oblong (or Geometrical Mean) the Sums make the Series of the Squares of all odd Numbers after 1; thus $1+4+4=9$, the Square of 3; also, $4+9+12=25$, the Square of 5.

Squares	1 .	4 .	9 .	16 .	25
Oblongs		2 .	6 .	12 .	20
Squares		9 .	25 .	49 .	81
Roots		3 .	5 .	7 .	9

Demon. Let n, $n+1$, be two adjacent Numbers in the natural Series, their Squares are nn, $nn+2n+1$, and the Sum of these two is $2nn+2n+1$; again, the interjacent Oblong being the Geometrical Mean, is the Product of the Roots $n\times\overline{n+1}=nn+n$, the Double of which is $2nn+2n$, which added to $2nn+2n+1$, the Sum is $4nn+4n+1$; and this is the Square of $2n+1$, an odd Number, because $2n$ is even; but $2n+1$ being in the 1st Step 3, and n encreasing gradually by 1, therefore $2n$ encreases gradually by 2, and so also must $2n+1$; consequently the several Values of $2n+1$ are the Terms of the odd Series, 3, 5, 7, &c.

Theorem XXXIII.

To the Product of every two adjacent Oblongs add the interjacent Square, the Sums make the Series of Biquadrates, or 4th Powers, of the natural Series after 1; thus, $2\times6+4=16$, the 4th Power of 4; also $6\times12+9=81$, the 4th Power of 3.

Squares	1 .	4 .	9 .	16 .	25	
Oblongs		2 .	6 .	12 .	20 .	30
Biqua.			16 .	81 .	256 .	15000
Roots			2 .	3 .	4 .	

Demon. $n-1$, n, $n+1$, express any 3 adjacent Numbers in the natural Series, and $n\times\overline{n-1}=nn-n$. also $n\times\overline{n+1}=nn+n$, two adjacent Oblongs, and their Product is n^4-n^2, to which add nn, the Sum is n^4; but in the 1st Case n is $=2$, and it encreases gradually in all the following Steps by 1; whence the Truth proposed is clear.

Theorem XXXIV.

The Product of two adjacent Oblongs is an Oblong, whose greater Side is the Square of the lesser Side of the greater Oblong multiplied, (or of the greater Side of the lesser Oblong.)

Demon. $n-1$, n, $n+1$ being 3 adjacent Numbers, then are $\overline{n-1}\times n=nn-n$, and $n\times\overline{n+1}=nn+n$, two adjacent Oblongs; and their Product is $n^4-n^2=\overline{n^2-1}\times n^2$, which is an Oblong whose greater Side is nn, the Square of n, the lesser Side of the greater, and greater Side of the lesser Oblong.

Or thus, Let a, b, c, be 3 Numbers differing by 1, then are ab, bc, two adjacent Oblongs, and their Product is $ab\times bc=ac\times bb$; but from the Nature of Arithmetical Progressions, when the common Difference is 1, then is $ac=bb-1$, for a, b, c may be repre-

represented thus, $a, a+1, a+2$. then is $a\times\overline{a+2}=a^2+2a$. and $\overline{a+1}^2=a^2+2a+1$, whence $a^2+2a=a^2+2a+1-1$, that is, $a^2=bb-1$, therefore $ac\times bb$ is an Oblong.

PROBLEM IV.

One Oblong given, to find another such, that the two admit of one Geometrical Mean, *i. e.* such that their Product is a square Number.

Rule. Multiply the given Number by 4, and its Square by 16 (the Square of 4) the Sum of these Products is the Number sought. Thus, the given Oblong being n, that sought is $16n^2+4n$.

DEMON. $16n^2+4n$ is $=4n\times\overline{4n+1}$, which is plainly Oblong; and to shew that n, $16n^2+4n$ admit one Geometrical Mean; or that $n\times 16n^2+4n$ $(=16n^3+4n^2)$ is a square Number; let us suppose $n=ab$, and $a=b+1$; then is $4n=4ab=2a\times 2b$; but since $a=b+1$, then is $a-b=1$, therefore $2a-2b=2$; and an Arithmetical Mean betwixt $2a$ and $2b$ is $a+b$; so that $2a-\overline{a+b}=1$ $(=a-b.)$ Then also is $\overline{a+b}^2=2a\times 2b+1=4ab+1$ (by taking $2a+1=a+b$, and $a+2=2b$) wherefore $4ab\times\overline{a+b}^2$ is an Oblong. Also $4ab\times 4ab\times\overline{a+b}^2$ is a Square, whose Root is $4ab\times\overline{a+b}$, so that $4ab\times\overline{a+b}^2$ is the Number sought, when $4ab$ is the Number given: But $4ab=4n$, and $\overline{a+b}^2=4ab+1$ $=4n+1$, therefore $4ab\times\overline{a+b}^2=4n\times\overline{4n+1}=16n^2+4n$.

SCHOL. As to the Invention of this Rule, it may be traced in this Manner: The Sides of a given Oblong being a, b, then, 1°. To find a Number which multiplied into ab will produce a Square; it is obvious that if we multiply ab into any square Number, as nn, the Product $abnn$, is a Number which multiplied into ab produces a Square, *viz.* $\overline{ab}^2\times n^2=\overline{abn}^2$. But then, 2°. The Question is, whether $ab\times nn$ be also an Oblong; which depends upon the Choice of nn; as to which, it is plain in the first Place, that ab, nn, cannot be Sides of an Oblong; for, by the Nature of Oblongs, and their Connection with Squares, the next Square to ab is $ab-b$, or $ab+a$ (Vid. *Theorem* XXVIII. *Cor.* I.) either of which has a greater Difference from ab than 1; wherefore if nn is the greater Side of an Oblong, the other must be greater than ab: Also, in order that the Product of that Oblong by ab may be a Square, the other Side of the Oblong sought must be the Product of ab by some square Number, which if we suppose to be xx, then the two Sides of the Oblong are $abxx=ax\times bx$, and nn. But 3°. The Question still remains, What Numbers we shall chuse for x and n, so as $ax\times bx$, and nn, be Sides of an Oblong? In order to which it may readily occur, that if 3 Numbers are $\div l$, differing by 1; then the Product of the Extremes, and Square of the Mean differ by 1, [as in the Demonstration of *Theor.* XXXIV. we see $ac=bb-1$] and so are Sides of an Oblong; wherefore it follows, that in the present Case, ax, n, bx, must be $\div l$ differing by 1; and consequently x must be 2, because $a-b=1$, and hence $2a-2b=2$; so that betwixt $2a$, $2b$ there is one Arithmetical Mean in Integers, *viz.* $a+b$, the common Difference being 1; wherefore it's plain, that n must be an Arithmetical Mean betwixt $2a$ and $2b$, *i. e.* $n=a+b$.

From this Investigation the Rule may be expressed in this Manner, *viz.* Take the given Oblong, *viz.* ab, and multiply it by $a+b$ $(=n)$ the Sum of the Sides (which is the Arithmetical Mean betwixt the Doubles of the Sides, *viz.* $2a$, $2b$,) the Square of the Product is the Oblong sought, $\overline{abn}^2$. But observe, that this Rule requires that the Sides of the given Oblong be known, whereas the former Rule requires only the Oblong it self.

PROBLEM V.

To find 3 square Numbers in Arithmetical Progression, *i. e.* that the middle one exceed the least as much as the greatest exceeds the middle; thus, we are to find a^2, b^2, c^2, such that $a^2-b^2=b^2-c^2$.

Rule.

Rule. Take any two Square Numbers, which call c^2, d^2, and let them be such that $2c^2$ be greater than d^2; then is the Difference, *viz.* $2c^2-d^2$, the lesser of the Roots sought; to the Sum of the same Numbers, *viz.* $2c^2+d^2$, add $2dc$, and the Total $2c^2+dd+2dc$, is the next greater Root sought. Again, to this last Root add also $2dc$, and the Sum $2c^2+dd+4dc$ is the greatest of the Roots sought.

Exam. Take $c=2$, and $d=1$, then is $c^2=4$ and $d^2=1$: Again, $2c^2-d^2=8-1=7$, the lesser Root. To the Sum of $8+1=9$ add $2dc=4$, the Total is 13, the next Root. *Lastly*, To this Root 13 add $2dc=4$, the Sum 17 is the other Root sought: For $7\times7=49$, $13\times13=169$, and $17\times17=289$; and $289-169=169-49=120$.

Demonstration and Investigation of this Rule.

If you take the 3 Numbers $2c^2-d^2$; $2c^2+d^2+2dc$; $2c^2+d^2+4dc$, and find their Squares by the common Operations; then it will be found to be in Arithmetical Progression; but the Truth of this we shall see by the following *Investigation.*

Suppose the 3 Roots sought represented thus, *viz.* a, $a+b$, $a+b+c$; their Squares are a^2; a^2+b^2+2ab; $a^2+b^2+c^2+2ab+2ac+2bc$; the Difference of the 1st and 2d is b^2+2ab, and of the 2d and 3d is $c^2+2ac+2bc$; and these two Differences are, by Supposition, equal, *i. e.* $b^2+2ab=c^2+2ac+2bc$; whence this Proportion is evident, *viz.* $b+2a : c+2a+2b :: c : b$, but the 2d Term is greater than the 1st, and therefore so is b greater than c; and consequently $2ab$ is greater than $2ac$. In the next Place then, from each of these Equals, *viz.* $b^2+2ab=c^2+2ac+2bc$, take b^2+2ac, the Remainders are equal, *viz.* $2ab-2ac=c^2+2bc-b^2$; divide both these by $2b-2c$, or $2\times\overline{b-c}$, the Quotes are equal, *viz.* $a=\frac{c^2+2bc-b^2}{2b-2c}$, which is the lesser Root sought. But now to make this a possible Solution, it's manifest, that c^2+2bc must be greater than b^2; and if we chuse b and c, so as they have this Condition, and b also be greater than c, it's certain from what is shewn, that we have all the 3 Roots sought, *viz.* a, $a+b$, $a+b+c$. What remains then is to shew, how to chuse b, c, with these Conditions, in order to which suppose $b=c+d$, then is $b^2=c^2+d^2+2cd$, $bc=c^2+cd$, and $2bc=2c^2+2cd$; hence $c^2+2bc-b^2=c^2+2c^2+2cd-c^2-d^2-2cd=2c^2-d^2$; and because $b-c=d$, therefore $2b-2c=2d$, so that $a=\frac{c^2+2bc-b^2}{2b-2c}=\frac{2c^2-d^2}{2d}$, wherein $2c^2$ must be greater than d^2; and to chuse c, d so, has in it no manner of difficulty; for it's easily done, whether we take c greater or lesser than d; wherefore having assumed c, d according to the Conditions mentioned, we have all the 3 Roots, for $a=\frac{2c^2-d^2}{2d}$, then because $b=c+d$, therefore $a+b=\frac{2c^2-d^2}{2d}+c+d=\frac{2c^2-d^2+2dc+2d^2}{2d}=\frac{2c^2+d^2+2dc}{2d}$; also, $a+b+c=\frac{2c^2+d^2+2dc}{2d}+c=\frac{2c^2+d^2+2dc+2dc}{2d}=\frac{2c^2+d^2+4dc}{2d}$. In the last Place, since the Squares of these fractionally expressed Roots, are in Arithmetical Progression, so must the Squares of the Numerators be, because the Denominators are the same; but these Numerators are the same Expressions as in the Rule, which is therefore demonstrated.

Schol. If in all Cases we make $d=1$, then the 3 Roots are, $2c^2-1$; $2c^2+2c+1$; $2c^2+4c+1$; that is, assume any Number c, and from double its Square take 1, the Remainder is the lesser Root sought; to double its Square add double the Root, and 1, the Sum is the next Root sought; then to double its Square add 4 Times its Root, and 1 (*i. e.* to the preceding Root found add double the assumed Number) the Sum is the greatest of the Roots sought.

LEMMA.

The Difference of two Integral Squares is either some odd Number greater than 1; or, it is an even Number greater than 4, and also a Multiple of 4.

DEMON. (1°.) If it's an odd Number it must be greater than 1; this is shewn in *Cor.* 4. *Theor.* IX. (2°.) If it's an even Number it must be greater than 4, as is shewn in *Cor.* 2d and 4th. *Theor.* IX. But again, it must be a Multiple of 4, which is demonstrated thus, by *Coroll.* 1. *Theor.* IX. The Difference of any two Integral Squares is either some one, or the Sum of some two or more Terms adjacent in the odd Series, whereof the lesser is greater than 1. If it's one of these Terms, or the Sum of an odd Number of them, then it is an odd Number; and if it is the Sum of an even Number of Terms, then it is an even Number; and I say all these are Multiples of 4. For any two adjacent Terms in that Series, whereof the lesser is above 1, may be represented thus, $2a+1$, and $2a+3$, whose Sum is $4a+4=4\times\overline{a+1}$; which shews the Truth proposed, if the Squares differ by the Sum of any two adjacent Terms above 1. If the Difference is the Sum of any other even Number of Terms, then the Sum of every Pair of them at equal Distance from the middle Pair, is equal to the Sum of the middle Pair, because they are in Arithmetical Progression; but the Sum of the middle Pair being adjacent Terms, is a Multiple of 4, consequently so is the Sum of each of the other Pairs. And hence, *Lastly*, The total Sum of all the Pairs must be a Multiple of 4, since each of the Parts, *i. e.* each of the Pairs is so.

PROBLEM VI.

Having any Integral Number, to find two Integral Squares, whose Difference is equal to that given Number.

Rule. The given Number being an odd Number greater than 1, or an even Number greater than 4, and measurable by 4 (as required in the preceding *Lemma*) take any two different Integers, whose Product is equal to the given Number, *i. e.* any two of its reciprocal Measures [and mind, that if the given Number is odd, we admit of 1 for a Measure, whose reciprocal Measure is the given Number it self; but if the given Number is even, the reciprocal Measures must be both even] then the half Sum and half Difference of these two Numbers, are the Roots of two Squares, whose Difference is the given Number; and thus by taking every Pair of reciprocal Measures of the given Number, you'll find all the possible Solutions of the Problem.

Exam. 1. Given $15=1\times15=3\times5$; there are two Solutions, *viz.* $\frac{15-1}{2}=7$, and $\frac{15+1}{2}=8$; or $\frac{5-3}{2}=1$; and $\frac{5+3}{2}=4$. For $7\times7=49$, $8\times8=64$, and $64-49=15$; also, $4\times4=16$ and $16-1=15$.

Exam. 2. Given $105=1\times105=3\times35=5\times21=7\times15$, which make 4 Solutions, *viz.* $\frac{105-1}{2}=52$, and $\frac{105+1}{2}=53$, or $\frac{35-3}{2}=16$, and $\frac{35+3}{2}=19$, or $\frac{21-5}{2}=8$, and $\frac{21+5}{2}=13$, or $\frac{15-7}{2}=4$, and $\frac{15+7}{2}=11$.

Exam. 3. Given $20=2\times10$; whence we have this one Solution $\frac{10-2}{2}=4$, and $\frac{10+2}{2}=6$; no other Pair of the reciprocal Measures of 20 being both even Numbers.

Exam.

Exam. 4. Given, $112=2\times56, =4\times28, =8\times14$; whence we have three Solutions, *viz.* 27 and 29, or 12 and 16, or 3 and 11.

DEMONSTRATION.

1. Let the given Number be called D, and any of its reciprocal Meaſures a, n, that is, $D=an$; then are $\frac{a-n}{2}$, $\frac{a+n}{2}$, two Roots, whoſe Squares differ by $D=an$; for theſe Squares are $\frac{aa+2an+nn}{4}$ and $\frac{aa-2an+nn}{4}$, whoſe Difference is plainly an.

2. Again, it is evident that as a, n, are ſuppoſed to be qualified, $\frac{a-n}{2}$, $\frac{a+n}{2}$, will be both Integers: For D being an odd Number, all its Meaſures, as, a, n, are odd Numbers, and ſo $a-n$, $a+n$, are both even (as explained in *Ch.* I. §.4.) conſequently $\frac{a-n}{2}$, $\frac{a+n}{2}$, are both Integers; if D is even, a, n, muſt both be even, for if the one is even, and the other odd, the Sum and Difference are both odd, and ſo have not an Integral half; but a, n, being both even, the Sum and Difference are both even, and conſequently their Half is an Integer.

3. Thus it is demonſtrated that all the Numbers found by this Rule are true Solutions of the Problem; but obſerve, that it does not from hence follow, that there can be no other Solutions than theſe, *i. e.* if $D=x^2-y^2$ (two Integral Squares) it remains to be demonſtrated, that x, y, muſt be one of them the half Sum, and the other the half Difference of ſome two Integers, a, n, ſuch that $an=D$ (the given Number) which I thus demonſtrate.

Let x be the greater Root ſought, the other, y, may be expreſſed $x-n$, whoſe Square is $x^2-2xn+n^2$; hence $x^2-\overline{x-n}^2=2xn-n^2=D$, and dividing both by n, it is $2x-n=\frac{D}{n}$, which muſt be an Integer, becauſe x, n, are ſuppoſed to be ſo, *i. e.* n meaſures D. Suppoſe then that $\frac{D}{n}=a$ (ſo that $D=an$) then is $2x-n=a$, and $2x=a+n$, and $x=\frac{a+n}{2}$, which is the one Root; alſo $y=x-n=\frac{a+n}{2}-n=\frac{a-n}{2}$, the other Root, according to the Rule.

SCHOLIUM I.

In the preceding Demonſtration you have alſo the *Inveſtigation* of this Rule after that Method which is called Analytical, by ſuppoſing x, and $x-n$, the two Roots ſought. And we have alſo a Demonſtration of the Rule, as a Solution of the Problem taken more univerſally, without reſpect to Integers; for whether D be Integral or Fractional, any two Numbers, a, n, ſuch that $an=D$, make $\frac{a-n}{2}$, $\frac{a+n}{2}$, a true Solution; and all the poſſible Solutions are got in this Method, becauſe x and $x-n$ may repreſent all the poſſible Roots.

But there are other Methods whereby this Rule might be diſcovered, of the Kind we call Synthetical, wherein we proceed from ſome known Truth; and theſe being alſo worth knowing, I ſhall explain them.

Second Method of Investigating the preceding Rule.

This Rule might be owing to the Consideration of the Squares of a Binomial and Residual Root. Thus, having observed that the Square of $a+n$ is $aa+2an+nn$, and the Square of $a-n=aa-2an+nn$; also, that the Difference of these two Squares is $4an$; it was obvious to conclude, that if the Difference of two Squares was a Multiple of 4, it may be expressed by $4b$, and b be expressed $a\times n$, (a, n being any two Numbers, whose Product is $=b$.) Again, it might easily occur, that though the Difference of every two Squares is not a Multiple of 4, yet the Difference of any two Squares may be represented by the Product of two Numbers, as an; and then, considering that $\frac{4an}{4}=an$; which being the 4th Part of the preceding Difference $4an$ ($=4b$) therefore it must be the Difference of the 4th Parts of the preceding Squares, *viz.* of $\frac{aa+2an+nn}{4}$ and $\frac{aa-2an+nn}{4}$, whose Roots are $\frac{a+n}{2}$, $\frac{a-n}{2}$; whence an universal Rule is discovered for finding two Squares, whose Difference is any given Number, provided always that a, n, be different Numbers, for otherwise $a-n$ is nothing. What other Conditions are necessary, upon Supposition that D is Integer, are also obvious, *viz.* that a, n be both Integers, and both odd, or both even.

Third Method of Investigating the preceding Rule.

The last Method seems to be rather by Chance, than that one is naturally directed to the Consideration of Binomial and Residual Squares, as a first Principle that should lead to the Solution of the Problem.

But after the Invention of *Theor.* IX. we have a more easy and natural Principle afforded us for the Solution of the Problem, as limited to Integers; which, though tedious, yet is very curious and not difficult; thus,

By *Coroll.* 1. *Theor.* IX. The Difference of any two Integral Squares is either some one, or the Sum of some 2 or more Terms of the odd Series, 1 . 3 . 5 . *&c.* and the Roots of these Squares are the Indexes of the Places of the greater Term, and of that next below the lesser of these Terms, whose Sum is equal to the given Number. It is hence evident, that we have no more to do, but from the given Number, and the known Properties of an Arithmetical Progression, to seek an Expression in known Numbers for these Roots; and all the Variety of them that have the same Difference; and this we must distribute into 2 Cases.

Case I. The given Number being an odd Number greater than 1. In this Case the Problem has at least one Solution; for D being a Term of the odd Series, whose Place call n, then is D the Difference of two Squares, whose Roots are n and $n-1$; and these being expressed according to the Rules of Arithmetical Progressions, are $n=\frac{D+1}{2}$, and $n-1=\frac{D-1}{2}$; for if the greater Extreme of an Arithmetical Series is l, the lesser a, and the common Difference d, then is the Number of Terms $n=\frac{l-a+d}{d}$. But in the present Case $l=D$, $a=1$, and $d=2$; hence $n=\frac{D-1+2}{2}=\frac{D+1}{2}$, and $n-1=\frac{D+1}{2}-1=\frac{D-1}{2}$; and these Roots $\frac{D-1}{2}$, $\frac{D+1}{2}$, are manifestly two Examples of the preceding Rule, because $1\times D=D$.

Again,

Again, To find if there can be any other Solutions, and what they are: Consider, That if there is another Solution, then, by *Coroll.* 1. *Theor.* IX. the given odd Number, D, must be equal to the Sum of some odd Number of Terms greater than 1, standing next together in the odd Series, and whereof also the least Term cannot be 1: And in this Case the Place of (or the Number of Terms from 1, to) the greatest of these Terms, and the Place next below the lesser of these Terms, are two Roots, whose Squares have the given Difference (which is the Sum of all these Terms.) Wherefore it's plain, That as many different odd Numbers of Terms as there are, whose Sums are each equal to D (of which the least is greater than 1) so many, and no more, Solutions has the Problem: And to discover these, consider *again*, That from the Nature of Arithmetical Progressions, when the Number of Terms is odd, it's an *aliquot* Part of the Sum; for the Sum being D, and Number of Terms n, the middle Term is $\frac{D}{n}$; which being, by Supposition, an Integer, n must measure, or be an *aliquot* Part of D; whereby it's evident, that as there can be no Solution of the Problem but one for every odd Number of Terms of the odd Series, whose Sum is $=D$, and whereof the least Term is greater than 1; so, because that Number of Terms is also an *aliquot* Part of D, it follows, that there cannot be more Solutions than there are different Measures of D. But yet again, I say that all the Solutions amount only to one for every Pair of Reciprocal Measures (and not one for every Measure:) And to demonstrate this, and also find the general Expression for them all, agreeable to the Rule, we proceed *thus*; D being the Sum of an odd Number, n, of Terms of the odd Series, $\frac{D}{n}$ is the middle Term; and then having $\frac{D}{n}$, the middle Term, and 2, the common Difference, we may find the Extremes thus; Suppose the middle Term, $\frac{D}{n}$, to be a lesser Extreme, with respect to the greater Extreme sought (which in general call l) and a greater, with respect to the lesser Extreme sought (which call A;) then, since the Number of Terms, from A to l, is n, the Number of Terms less 1, from A to $\frac{D}{n}$, and from $\frac{D}{n}$ to l, is $\frac{n-1}{2}$; and hence, by the common Rules, we find $l=\frac{D}{n}+2\times\frac{n-1}{2}=\frac{D}{n}+n-1$; and $A=\frac{D}{n}-2\times\frac{n-1}{2}=\frac{D}{n}-n+1$; then the Places of these Terms in the odd Series, are $\frac{l+1}{2}$, the Place of l; which is also the Root of the greater Square sought; and $\frac{A+1}{2}$, the Place of A; from which take 1, the Remainder is $\frac{A-1}{2}$, the lesser Root sought; now putting instead of l, and A, their Equals, $\frac{D}{n}+n-1$ $(=l,)$ and $\frac{D}{n}+1-n$ $(=A)$ these Roots are $\frac{D+nn}{2n}$, and $\frac{D-nn}{2n}$; for $l+1=\frac{D}{n}+n=\frac{D+nn}{n}$; which divided by 2, Quotes $\frac{D+nn}{2n}$; and $A-1=\frac{D}{n}-n=\frac{D-nn}{n}$; which divided by 2, Quotes $\frac{D-nn}{2n}$. Again, Since n measures D, suppose $\frac{D}{n}=a$, then is $D=an$; and putting an for

D,

D, the Roots are $\frac{an+nn}{2n}$, $\frac{an-nn}{2n}$; which by equal Division of Numerator and Denominator, by n, are $\frac{a+n}{2}$, $\frac{a-n}{2}$; which Expressions comprehend also the 1st Solution; wherein n being $=1$, D is $=a$: And hence $\frac{a-n}{2}$, $\frac{a+n}{2}$, are $=\frac{D-1}{2}$, $\frac{D+1}{2}$, the Rules given for that Case; so that D being $=an$, any two reciprocal Measures of D, all the possible Roots which solve the Problem, are expressed in general, by $\frac{a-n}{2}$, $\frac{a+n}{2}$, according to the preceding Rule: Then, because n must be less than a, to make $a-n$ possible; therefore of two reciprocal Measures of D, there can be a Solution only for the lesser of them, taken as a Number of Terms, whose Sum is $=$D. And thus is the whole Investigation and Demonstration finished, when D is an odd Number.

Case II. The given Number, D, being an even Number (greater than 4, and measurable by 4) then the Problem has, at least, one Solution; for by Supposition, $\frac{D}{2}$ is an even Number; therefore $\frac{D}{2}+1$, $\frac{D}{2}-1$, are two odd Numbers; which differing by 2, shews that they stand next together in the odd Series; and their Sum being $=$ to D, therefore the Places of $\frac{D}{2}+1$, and of the Term next below $\frac{D}{2}-1$, are two Roots, whose Squares differ by D. Now these Places are found by the common Rules to be $\frac{D+4}{4}$, and $\frac{D-4}{4}$. Again, Since D is a Multiple of 4, let it be $=4d=2\times 2d$; then $\frac{D+4}{4}=\frac{4d+4}{4}=\frac{2d+2}{2}$ $(=d+1)$ and $\frac{D-4}{4}=\frac{4d-4}{4}=\frac{2d-2}{2}$ $(=d-1)$ which shews us plainly that this is an Example of the preceding Rule.

Again, To find if there are any other Solutions, and what they are. Consider, That if there is another Solution, then D must be the Sum of an even Number of Terms greater than 2, standing next together in the odd Series, and whereof the least is not 1, (by *Cor.* 1. *Th.* IX.) and in this Case the Places of the greatest of these Terms, and of that next before the least, are 2 Roots, whose Squares differ by D, the Sum of all these Terms; wherefore it's plain, that as many different Numbers of Terms as there are, whose Sums are each $=$D, and of which the least Term is greater than 1, so many, and no more, Solutions has the Problem: And to discover these, consider, *again*, That from the Nature of a Progression of odd Numbers, if the Number of Terms is even, it's an *aliquot* Part or Measure of the Sum; for D being the Sum, n the Number of Terms, and the Extremes, A, L, then is D $=\overline{A+l}\times\frac{n}{2}$. Hence $A+l=D\div\frac{n}{2}=\frac{2D}{n}$, and $\frac{A+l}{2}=\frac{D}{n}$; but A, l, are both odd, and so their Sum is even, *i. e.* $\frac{A+l}{2}$, is an Integer; consequently so is $\frac{D}{n}$, or n measures D: Hence, if there is any Number of Terms of the odd Series, whose Sum is $=$D, that Number measures D; whereby it's evident there can be no Solution of the Problem, but one, for every even Number of Terms, whose Sum is D, and whereof the least Term is greater than 1; and because that Number of Terms must measure D, it follows, that there cannot be more Solutions than there are even Measures of D: But yet again, I say, that there cannot be more Solutions than the Number of reciprocal Measures of D, or one for each Pair, which are both even Numbers: And

And to demonstrate this, and find the general Expression for them all, agreeable to the Rule, we proceed thus; D being the Sum of an even Number, n, of Terms of the odd Series, whose common Difference is 2, then $\frac{D}{n}-1$, $\frac{D}{n}+1$, express the two middle Terms; for the Sum of the two middle Terms is equal to the Sum of the Extremes, $A+l$, which is equal to $\frac{2D}{n}$; because $\overline{A+l}\times\frac{n}{2}=D$, and hence $A+l=\frac{2D}{n}$; therefore the Sum of the two middle Terms is also $=\frac{2D}{n}$. But it is a known Truth, that the half Sum, more or less the half Difference of two Numbers, is equal to the greater or lesser of them; wherefore $\frac{2D}{n}$ being the Sum of the middle Terms, $\frac{D}{n}$ is the half Sum; also 2 being their Difference, 1 is the half Difference; therefore $\frac{D}{n}+1$ is the greater, and $\frac{D}{n}-1$ the lesser middle Term: From which, *again*, we may find the two Extremes, A, l, *thus*; Suppose the greater middle Term, $\frac{D}{n}+1$, to be a lesser Extreme, with respect to l; and $\frac{D}{n}-1$ to be a greater Extreme, with respect to A; then the Number of Terms, from A to l, being n, that from A to $\frac{D}{n}-1$, and from $\frac{D}{n}+1$ to l, will be $\frac{n}{2}$; from which take 1, the Remainder is $\frac{n}{2}-1=\frac{n-2}{2}$; and from the common Rules, l is $=\frac{D}{n}+1+2\times\frac{n-2}{2}=\frac{D}{n}+n-1$; also, $A=\frac{D}{n}-1-2\times\frac{n-2}{2}=\frac{D}{n}+1-n$. Now these being the same Expressions, as for l and A, in the 1st Case; all the rest of the Investigation is the same as there; whereby the two Roots sought are $\frac{a-n}{2}$, $\frac{a+n}{2}$; which comprehend also the 1st Solution, wherein $n=2$, $D=4d=2d\times 2$; whence $2d=a$, and $\frac{a-n}{2}=\frac{2d-2}{2}=d-1$; also $\frac{a+n}{2}=\frac{2d+2}{2}=d+1$, which are the Rules of that first Solution: So that D being $=an$ (any two reciprocal Measures of D) all the Solutions of the Problem are universally expressed by $\frac{a-n}{2}$, $\frac{a+n}{2}$, upon Condition also that a, n, be both even Numbers; for else the Roots cannot be Integers: They must also be different Numbers; else $a-n=o$: Whence, in the last Place it is clear that all the Solutions amount only to one for every Pair of the reciprocal Measures of D, which are both even Numbers.

SCHOLIUM II.

In the preceding Problem, the given Difference being called D, will, in some Solutions, be less than $\frac{a-n}{2}$ squar'd, and in some greater; and indeed, in some Values of D, there will be no Solution in which D will be less than $\frac{a-n}{2}$ squared: And in some

Cases

Cases there will be no Solution, wherein D is greater than $\frac{a-n}{2}$ squar'd. In other Cases you will find Solutions of both Kinds. But I shall also explain these Things more particularly.

1*st*. If D is an even Number, then (1.) If it's less than 24, it must be 8, 12, 16, or 20; for no other even Number less than 24, is a Multiple of 4: And in all these there is but one Solution, *viz.* Where 2 is one of the reciprocal Measures; because no other Pair of reciprocal Measures of any of these Numbers, are both even Numbers; and the lesser Roots in these several Examples are $\frac{4-2}{2}$, $\frac{6-2}{2}$, $\frac{8-2}{2}$, $\frac{10-2}{2}$ (*viz.* when $8=4\times2$, $12=6\times2$, $16=8\times2$, $20=10\times2$, are the Values of D) and the Squares of these are all less than D, or 24; for they are, 1, 4, 9, 16. (2.) If D is 24, or greater, then in some Values of it there will be but one Solution; and in that, D ($=a\times n$) will be less than $\frac{a-n}{2}$ squared: And such are all these Values of D, whose 4th Part is an odd Number; for since D must be a Multiple of 4, let it be $=4d$; if its 4th Part, d, is an odd Number, it follows, that it can have no Pair of even Numbers for its reciprocal Measures, but 2 and $2d$; so that there is but this one Solution: And here the Roots are $\frac{2d-2}{2}$, $\frac{2d+2}{2}$, which are equal to $d-1$, and $d+1$; whose Squares are both greater than D. That $\overline{d-1}^2$ is greater than $D=4d$, is thus proved; $\overline{d-1}^2=dd-2d+1$; add $2d$ both to this Square and to $4d$; the Sums are $dd+1$, and $6d$; divide both by d, and the Quotes are $d+\frac{1}{d}$, and 6: But $4d$ being equal to 24, or greater, it follows, that D must be $=6$, or greater; and consequently $d+\frac{1}{d}$ is greater than 6; and hence $dd+1$ is greater than $6\times d$; also $dd-2d+1$, greater than $6\times d-2d=4d$; that is, $\overline{d-1}^2$ greater than $D=4d$.

In all other Values of D (greater than 24, and also an even Number) there will be, at least, two Solutions; for we suppose now, that the 4th Part of D is an even Number; and therefore D may be represented, $4\times2d=2\times4d$, which make two Solutions; and according as d is variously compounded, so will there be a Variety of other Solutions; in some of which D will be less, and in some greater, than the lesser of the two Squares.

2*d*. If D is an odd Number, then (1.) If it's a prime Number, there is but one Solution, and the Roots sought are $\frac{D-1}{2}$, and $\frac{D+1}{2}$. And here, if D is 3 or 5, it is greater than the lesser Square; for this, in these Cases, is 1, or 4. But if D is greater than 5, then it's always less than the lesser Square sought, *viz.* $\frac{D-1}{2}$ Square. For $\overline{D-1}^2=D^2-2D+1$, and $\frac{D-1}{2}$ Square, $=\frac{D^2-2D+1}{4}$; compare this with D, thus, multiply both by 4, and the Products are D^2-2D+1, $4D$; add $2D$ to both, the Sums are D^2+1, $6D$; divide both these by D, the Quotes are $D+\frac{1}{D}$,

6;

6; but D is ſuppoſed to be an odd Number greater than 5, that is, 7 at leaſt, and conſequently $D+\frac{1}{D}$ is greater than 6: And hence going backwards, D^2+1 is greater than 6D; alſo D^2+1-2D, greater than $6D-2D=4D$; and $\frac{D^2+1-2D}{4}$ greater than D, or D leſs than $\frac{D-1}{2}$ ſquar'd. (2.) If D is an odd compoſite Number, there will be ſeveral Solutions, according to the Variety of the Compoſition; and in all Caſes, there will be at leaſt, two Solutions, whereof the Roots of one will be $\frac{D-1}{2}$, $\frac{D+1}{2}$, both their Squares being greater than D; becauſe D is greater than 5; as already ſhewn.

3*d*. Another Thing to be obſerved here, is, That having found all the reciprocal Meaſures of D; if you begin with that Pair, which conſiſts of the leaſt and greateſt Meaſures of D, *i. e.* that Pair which have the greateſt Difference; or alſo with that Pair which have the leaſt Difference; and ſo proceed in Order: Then, if the firſt Solution makes $\frac{a-n}{2}$ ſquared, leſs or greater than D, go on till you have a Solution making $\frac{a-n}{2}$ ſquared, contrarily greater or leſs than D; and after this, all the reſt will be of the ſame Kind. The Reaſon is plain; for let the reciprocal meaſures of D be repreſented as in the Margin;

$$\begin{array}{c} a\ .\ b\ .\ c\ .\ \&c. \\ A\ .\ B\ .\ C\ .\ \&c. \\ \hline aA=bB=cC=\&c.=D \end{array}$$

wherefore if a is the leaſt, and A the greateſt Meaſure of D, then it is plain that $A-a$ is greater than $B-b$, which is greater than $C-c$ &c. Therefore if $\frac{A-a}{2}$ ſquared is greater than D, then having proceeded till we find a Solution, wherein $\frac{x-y}{2}$ ſquared (x, y, being any Pair of reciprocal Meaſures of D) is leſs than D, all that follow will be ſo too; becauſe the Difference of the two reciprocal Meaſures grows ſtill leſs and leſs. Again, if $A-a$ is leſs than $B-b$, &c. (*i. e.* if a is the greateſt Meaſure of D, which is leſs than its reciprocal Meaſure) then if $\frac{A-a}{2}$ ſquared is leſs than D, having proceeded till we find a Solution, in which $\frac{x-y}{2}$ ſquared is greater than D, all after this will be ſo too; becauſe the Differences increaſe.

PROBLEM VII.

To find 3 Integral Squares (or 3 Integral Roots, whoſe Squares are) ſuch, that the greater is equal to the Sum of the two leſſer.

Rule. Aſſume any ſquare Number, N^2, greater than 4; and take any two of its reciprocal Meaſures, as a, n; then are $\frac{a-n}{2}$, $\frac{a+n}{2}$, the Roots of two Squares, which with N^2 ſolve the Problem.

Exam. 1. If we aſſume $9=1\times 9$; the 3 Squares ſought are 9, 16, 25.
Exam. 2. If we aſſume $64=2\times 32=4\times 16$; the 3 Squares ſought are 64, 225, 289, or 64, 36, 100.

DEMON. The univerfal Reafon of this is contained in the Demonftration of the preceding Problem; for the Difference of two Squares may be any odd Number greater than 1, or any even Number greater than 4, and meafurable by 4, (by the preceding *Lemma*) therefore it may be any Square Number greater than 4; becaufe any fuch Square is either an odd Number greater than 1, or it is an even Number meafurable by 4 (as all even Squares are, by *Theor.* XXIX. § 4. *Chap.* 1.)

SCHOLIUMS.

I. By this Rule, the Square affumed is always one of the two leffer, becaufe it is the Difference of the other two; and if the Problem be limited to this Condition, *viz.* That of the three Squares, the affumed (or given) one be the leaft, or alfo the middle one of the three, then it may be done thus:

(1°.) To make the affumed Square the leaft of the Three; if you affume N^2, an odd Square, then as it muft be greater than 1, fo the other 2 Roots will be $\frac{N^2+1}{2}$, and $\frac{N^2-1}{2}$, which will be greater than N; as is fhewn in the fecond Article of *Schol.* 2*d.* to the preceding *Problem*; where it's fhewn, that if D (an odd Number) is greater than 5, then $\frac{D-1}{2}$ fquared is greater than D, *i.e.* in the prefent Cafe, $\frac{N^2-1}{2}$ fquar'd is greater than N^2; for here N^2 is at leaft 9. Again, if you affume N^2 an even Number, as it muft be a Multiple of 4, in order to be the Difference of 2 Squares, fo it muft be the Multiple of it by fome Square Number, for elfe the Product of it by a Square 4 could not produce a Square Number (by *Coroll.* 2. *Theor.* II. *Book* III. *Chap.* I.) Suppofe now, that $N^2=4\times a^2=2\times 2a^2$, then are $\frac{2a^2-2}{2}$ $(=a^2-1)$ and $\frac{2a^2+2}{2}$ $(=a^2+1)$ two Roots, which folve the Problem in all Cafes wherein N^2 is greater than 16; as appears by the firft Article of *Schol.* 2. to the preceding Problem; for there it's fhewn, that when D (or in this Cafe N^2) is greater than 24 (as it muft be if it's a Square Number greater than 16) then it is lefs than $\overline{d-1}^2$, the leffer of the Squares fought (which is in this Cafe a^2-1 fquar'd.) But if $N^2=16$, this will not folve the Problem, becaufe then $4\times a^2$ $(=N^2)=16$, and confequently $a^2=4$, and $a^2-1=3$, lefs than $4=N$.

(2°.) To make the affumed Square Number the middle one of the Three; then find all its reciprocal Meafures, and take that Pair which have the leaft Difference, fuppofe them to be x, y; then if $\frac{x-y}{2}$ is lefs than N, the Problem is poffible, and $\frac{x-y}{2}$, $\frac{x+y}{2}$, are two Roots which folve it; but if $\frac{x-y}{2}$ is greater than N, the Problem is impoffible; for the Difference of all the other Pairs of reciprocal Meafures being, by Suppofition, greater than $x-y$, therefore $\frac{x-y}{2}$ will be lefs than the half of the Difference of thefe others; fo that $\frac{x-y}{2}$ being greater than N, and the half Difference of all the others being greater than $\frac{x-y}{2}$, they muft ftill be greater than N, and confequently N^2 is the leaft, and not the middle of the three Squares.

II. If

II. If the given Square Number is propoſed as the greateſt of the three, [which is the ſame Problem as propoſing to divide a given Integral Square into two other Integral Squares] the Limitation is ſtill more difficult; and the preceding Rule is of no uſe here; nor do I indeed know any Rule better or eaſier than taking all the Numbers leſs than N, and adding together the Squares of every one of them, whereby you'll find every Pair of Squares leſs than N^2, whoſe Sum is equal to N^2.

But in ſome Caſes the Impoſſibility of this Problem may be diſcovered without the Application of this tedious Rule; thus, divide the given Square by 4; and if the Remainder is 3, the Problem is impoſſible; the Reaſon of which you have in *Cor.* 4. *Theor.* XXIX. § 4. *Chap.* I. Yet obſerve, that though the Remainder is not 3, or if there is no Remainder, as in all even Squares, it does not follow that therefore the given Square is diviſible into two Squares.

THEOREM XXXV.

If 3 Numbers are ſuch that the Square of the greater is equal to the Sum of the Squares of the other two, the ſame will be true alſo of any the like Multiples, or *aliquot* Parts of theſe Numbers.

Exam. 3, 4, 5, are ſuch Numbers as propoſed; and ſo are their Doubles, 6, 8, 10, and their Triples, 9, 12, 15; as you'll find by Calculation.

DEMON. Let A, B, C, be ſuch that $C^2 = A^2 + B^2$, then are nA, nB, nC, ſuch alſo, *viz.* $\overline{nC}^2 = \overline{nA}^2 + \overline{nB}^2$, for $\overline{nC}^2 = n^2 \times C^2$, $\overline{nB}^2 = n^2 \times B^2$, $\overline{nA}^2 = n^2 \times A^2$; but $C^2 = A^2 + B^2$, therefore their Equimultiples are alſo equal, *viz.* $n^2C^2 = n^2A^2 + n^2B^2$. The ſame is true of like *aliquot* Parts, which is but the Reverſe of the former, ſince nA, nB, nC, may repreſent any 3 Numbers which have a like *aliquot* Part denominated by n.

COROLLARIES.

1. If it is propoſed to find 3 Integral Numbers, ſuch that the Square of the greater be equal to the Sum of the other two Squares, and ſuch too, that the Roots be in certain Ratio's to one another, then reduce theſe Ratios to a common Antecedent, *i. e.* find 3 Numbers, which taken in a Series, are gradually to one another in the propoſed Ratios (by *Probl.* I. *Book* IV. *Chap.* IV.) and if theſe 3 Numbers anſwer the Problem, you have done, otherwiſe the Problem is impoſſible; for if there are any other 3 Numbers in theſe Ratios that can ſolve the Problem, they are either the leaſt Terms of theſe Ratios, or like Multiples of them; if the leaſt Terms, then the like Multiples will alſo ſolve the Problem: If you ſay they are ſome like Multiples of the leaſt Terms, then alſo the leaſt Terms, which are like *aliquot* Parts of them, will ſolve it; but the 3 Numbers firſt found muſt alſo be either the leaſt Terms of the ſame Ratios, or like Multiples of them; and therefore if they ſolve not the Problem, it cannot be ſolved, becauſe if any Terms of theſe Ratios ſolve it, all Terms muſt ſolve it.

2. Having any 3 Numbers, ſuch that the Square of the greater is equal to the Sum of the Squares of the other two, we can find an infinite Number of other Examples of the ſame kind, by taking any like Multiples of the Numbers given.

THEOREM XXXVI.

If the Root of any Square Number is equal to the Sum of two Square Numbers, that Square is also equal to the Sum of two Square Numbers, whose Roots are, the Difference of the two Squares whose Sum the first Root is, and double the Product of their Roots.

Exam. $4+9=13$, and $13\times13=169=144+25$, two Squares, whose Roots are 12 ($=2\times2\times3$) and 5 ($=9-4$.)

DEMON. Take any two Squares, a^2, b^2, their Sum is a^2+b^2, and the Square of this Sum is $a^4+2a^2b^2+b^4=a^4-2a^2b^2+b^4+4a^2b^2$; also $a^4-2a^2b^2+b^4$ is a Square, whose Root is a^2-b^2; and $4a^2b^2$ is a Square, whose Root is $2ab$.

COROLL. Hence we learn how to find a Square Number, which is equal to the Sum of two Squares; or, another Rule for finding 3 Square Numbers, such that the greater is equal to the Sum of the two lesser. But *observe*, that though we can hereby find an infinite Number of Answers to this Problem, yet all the possible Solutions of the Problem cannot be found in this Method; because all the Examples of this Rule are of a particular Kind, *viz.* where the Root of the greatest Square is it self the Sum of two Squares; whereas there is an infinite Number of other Examples not of this Kind; thus, 9, 12, 15, are such Roots, and yet 15 is not the Sum of two Squares: But all the possible Examples of this Problem are to be found by *Probl.* VII. and therefore all the Examples found by this Rule must coincide with some of those found by that universal Rule; which Coincidence you may see thus: Let the assumed Root of a Square be $2ab$, its Square is $4a^2b^2=2a^2\times2b^2$, and by the Rule of *Probl.* VII. $\frac{2a^2-2b^2}{2}=a^2-b^2$, $\frac{2a^2+2b^2}{2}=a^2+b^2$, are two Roots which solve the Problem; and of which a^2+b^2, the greatest of the 3 Roots, is it self the Sum of two Squares, the Roots of the other two being the Difference of these two Squares, *viz.* a^2-b^2, and double the Product of their Roots, *viz.* $2ab$; agreeable to the present Theorem.

CHAP.

CHAP. III.

Of Infinite Series.

DEFINITION.

AN *Infinite Series* is a Series consisting of an Infinite Number of Terms, *i. e.* to the end of which 'tis impossible ever to come; so that let the Series be carried on to any assignable length, or number of Terms, it can be carried yet farther, without End or Limitation.

SCHOLIUM. A Number actually Infinite (*i. e.* all whose Units can be actually assign'd, and yet is without Limits) is a plain Contradiction to all our Ideas about Numbers; for whatever Number we can actually conceive, or have any proper Idea of, is always determinate and finite; so that a greater after it may be assign'd, and a greater after this, and so on, without a Possibility of ever coming to an end of the Addition or Encrease of Numbers assignable: Which Inexhaustibility, or endless Progression in the nature of Numbers, is all that we can distinctly understand by the *Infinity* of Number: And therefore, to say that the Number of any Things is Infinite, is not saying that we comprehend their Number, but indeed the contrary; the only Thing positive in this Proposition being this, *viz.* That the Number of these Things is greater than any Number which we can actually conceive and assign. But then, whether in Things that do really exist it can truly be said, that their Number is greater than any assignable Number; or, which is the same thing, That in the Numeration of their Units one after another 'tis impossible ever to come to an End; this, I say, is a Question about which there are different Opinions, with which we have no business in this place; for all that we are concern'd to know here is this certain Truth, That after one determinate Number, we can conceive a greater, and after this a greater, and so on without end. And therefore, whether the Number of any Things that do or can really exist all at once, can be such that it exceeds any determinable Number, or not, this is true, That of Things which exist, or are produced successively one after another, the Number may be made greater than any assignable one; because tho' the Number of Things thus produced that does actually exist at any time is *Finite*, yet it may be encreas'd without end. And this is the distinct and true Notion of the *Infinity of a Series*; *i. e.* of the Number of its Terms, as it is express'd in the *Definition.*

From hence again 'tis plain, That we cannot apply to an *Infinite Series* the common Notion of a Sum, *viz.* a Collection of several particular Numbers that are join'd and added together one after another, for this supposes that these Particulars are all known and determin'd; whereas the Terms of an *Infinite Series* cannot be all separately assign'd, there being no end in the numeration of its Parts, and therefore it can have no Sum in this Sense. But again, consider that the Idea of an *Infinite Series* consists of two Parts, *viz.* the Idea of something positive and determin'd, in so far as we conceive the *Series* to be actually carried on; and the Idea of an inexhaustible Remainder still behind, or an endless Addition of Terms that can be made to it one after another; which is as different from the Idea of a *Finite Series* as two Things can be: Hence we may conceive it as a Whole of its own Kind, which therefore may be said to have a total Value whether that be determinable, or not. Now in some *Infinite Series* this Value is finite or limited; *i. e.* a Number is assignable beyond which the Sum of no assignable Number of Terms of

the Series can ever reach, nor indeed ever be equal to it, yet may approach to it in such a manner as to want less than any assignable Difference; and this we may call the Value or Sum of the Series; not as being a Number found by the Common Method of Addition, but as being such a Limitation of the Value of the Series, taken in all its Infinite Capacity, that if it were possible to add them all one after another, the Sum would be equal to this Number.

Again: In other Series the Value has no Limitation; and we may express this by saying, *The Sum of the Series is Infinitely Great*; which indeed signifies no more than that it has no determinate and assignable Value; and, that the Series may be carried such a length as its Sum, so far, shall be greater than any given Number. In short, in the first Case we affirm there is a Sum, yet not a Sum taken in the common sense; in the other Case we plainly deny a Determinate Sum in any sense. What kind of Series have *Finite* or *Infinite* Sums in these senses, you'll learn in what follows.

THEOREM I.

In an *Infinite Series* of Numbers, encreasing by an equal Difference or Ratio [*i. e.* an Arithmetical or Geometrical encreasing Progression] from a given Number, a Term may be found greater than any assignable Number.

DEMON. 1°. If it's an Arithmetical Progression, let the Distance of any Term from the first be call'd n; the first, A; and the common Difference d; then any Term after the first is $A + nd$: And, that this may be found greater than any assign'd Number B, is thus prov'd: Suppose $B \div d = q$, then is $B = dq$. But whatever q is, since it is Finite, we can take a greater Finite Number; and therefore we may suppose or take n greater than q, so that nd will be greater than qd or B; and $A + nd$ greater than $A + qd$; but $qd = B$, therefore $A + nd$ is greater than $A + B$, and consequently yet greater than B.

2°. If it's a Geometrical Progression, the differences of its Terms make also a Geometrical Progression encreasing (*Theo.* 18, Ch. 3, B. 4.) But the thing propos'd being true, in case the Differences in a Series are equal, (as in an Arithmetical Progression) it must necessarily be so, and after a smaller number of Terms too, where the Differences do continually encrease (as in a Geometrical Progression). Or we may prove this independently of the other, thus: Let the first Term of a Geometrical Progression be A, the Ratio r, and the Distance of any Term from the 1st be n, then is that Term Ar^n. But we may take n greater than any assign'd Number; and it's plain that Ar^n will be yet much greater than that Number.

COROLLARY.

If the *Series* encrease by Differences that continually encrease, or by Ratio's that continually encrease, comparing each Term to the preceeding, it's manifest that the same thing must be true, as if the Differences or Ratio's continued equal.

THEOREM II.

In a *Series* decreasing *in infinitum* in a given Ratio, we can find a Term less than any assignable Fraction.

DEMON. The first Term being l, and the Ratio r, a whole or mix'd Number, the Series is $l : \frac{l}{r} : \frac{l}{r^2}$ &c. Wherein the Denominators continually encrease in the Ratio $1 : r$. Suppose then the assign'd Fraction is $\frac{a}{b}$, take $a : b : : l : m$, then is $\frac{a}{b} = \frac{l}{m}$.

But

But we can find a Power of r, as r^n, greater than any assignable Number m (by the last). Hence $\frac{t}{r^n}$ is less than $\frac{t}{m}$ or $\frac{a}{b}$.

COROLL.

If the Terms decrease, so as the Ratio's of each Term to the preceeding do also continually decrease, then the same thing is also true as when they continue equal.

SCHOLIUM. Some may possibly think these two Theorems might have pass'd for Axioms, because the Notion of a Series continually encreasing or decreasing, may seem to include them; but you'll find in what follows, that a Series may have its Terms continually encreasing, yet so that no one of them can ever be actually found equal to a certain assignable Number. And for a decreasing Series, tho' its Terms continually decrease, yet it may be so, that no Term can ever be actually found so little as a certain assignable Number: And here, to distinguish these different Kinds of encreasing and decreasing Series, we may call such as encrease or decrease above or below any assignable Number, A Series encreasing or decreasing *infinitely*; and such as encrease or decrease continually, yet so as never to reach a certain assignable Number, we may call them *Infinite Series*, encreasing or decreasing *limitedly*; and when we say in general *an Infinite Series*, that may be taken indifferently, for either kind.

THEOREM III.

The Sum of an *Infinite Series* of Numbers all equal, or encreasing continually, by whatever Differences or Ratio's, is infinitely-great; *i. e.* such a Series has no determinate Sum, but grows so as to exceed any assignable Number.

DEMON. 1°. If the Terms are all equal, as $A : A : A$, &c. then the Sum of any Finite Number of them is the Product of A by that Number, as An; but the greater n is, the greater is An; and we can take n greater than any assignable Number, therefore An will be yet greater than that assignable Number.

2°. Suppose the Series encreases continually, (whether it do so *infinitely* or *limitedly*) then its Sum must be infinitely great, because it would be so if the Terms continued all equal, and therefore will rather be so if they encrease. But if we suppose the Series encreases infinitely, either by equal Ratio's or Differences, or by encreasing Differences or Ratio's of each Term to the preceeding; then the Reason of the Sum's being Infinite will appear from the first Theorem; for in such Series a Term can be found greater than any assignable Number, and much more therefore the Sum of that and all the preceeding.

THEOREM IV.

The Sum of an *Infinite Series* of Numbers decreasing in the same Ratio is a *Finite Number*; equal to the Quote arising from the Division of the Product of the Ratio and first Term, by the Ratio less Unity; that is, the Sum of no assignable Number of Terms of the Series can ever be equal to that Quote; and yet no Number, less than it, is equal to the value of the Series, or to what we can actually determine in it; so that we can carry the Series so far, that the Sum shall want of this Quote less than any assignable Difference.

DEMON. To whatever assign'd Number of Terms the Series is carried, it is so far Finite; and if the greatest Term is l, the least A, and Ratio r, then the Sum is $S = \frac{rl - A}{r - 1}$ (*Prob.* 4, *Ch.* 3, *E.* 4.) Now, in a decreasing Series from l, the more Terms we actually raise, the last of them, A, becomes the lesser, and the lesser A be

$rl - A$ is the greater, and so also is $\frac{rl - A}{r - 1}$ (for the greater the Dividend with the same Divisor, the greater is the Quote): But $rl - A$ being still less than rl, therefore $\frac{rl - A}{r - 1}$ is still less than $\frac{rl}{r - 1}$ *i. e.* the Sum of any assignable Number of Terms of the Series is still less than the Quote mention'd, which is $\frac{rl}{r - 1}$, and this the First Part of the Theorem.

Again: The Series may be actually continued so far, that $\frac{rl - A}{r - 1}$ shall want of $\frac{rl}{r - 1}$ less than any assignable Difference; for, as the Series goes on, A becomes less and less in a certain Ratio, and so the Series may be actually continu'd till A becomes less than any assignable Number (*Theor.* II.) Now $\frac{rl}{r - 1} - \frac{rl - A}{r - 1} = \frac{A}{r - 1}$ (by the common Rules) and $\frac{A}{r - 1}$ is less than A; therefore let any Number assign'd be call'd N, we can carry the Series so far till the last Term A be less than N: And because $\frac{rl - A}{r - 1}$ wants of $\frac{rl}{r - 1}$ the Differ. $\frac{A}{r - 1}$, which is less than A, which is also less than N, therefore the second Part of the Theorem is also true, and $\frac{rl}{r - 1}$ is the true Value of the Series.

Scholium. 1. The Sense in which $\frac{rl}{r - 1}$ is call'd *The Sum of the Series*, has been sufficiently explain'd; to which however I have this to add, That whatever Consequences follow from the Supposition of $\frac{rl}{r - 1}$ being the true and adequate Value of the Series taken in all its *Infinite* Capacity, as if the whole were actually determin'd and added together, can never be the Occasion of any assignable Error in any Operation or Demonstration where it is used in that sense; because if you say it exceeds that adequate Value, yet it's demonstrated, that this Excess must be less than any assignable Difference, which is in effect no Difference, and so the consequent Error will be in effect no Error: For if any Error can happen from $\frac{rl}{r - 1}$ being greater than it ought to be to represent the complete Value of the *Infinite Series*, that Error depends upon the Excess of $\frac{rl}{r - 1}$ over that complete Value; but this Excess being unassignable, that consequent Error must be so too; because still the less the Excess is, the less will the Error be that depends upon it. And for this Reason we may justly enough look upon $\frac{rl}{r - 1}$ as expressing the adequate Value of the *Infinite Series*. But we are further satisfied of the Reasonableness of this, by finding in Fact, that a Finite Quantity does actually convert into an *Infinite Series*, which we have already seen in the Case of *Infinite Decimals*. For Example; $\frac{2}{3} = .6666$, &c. which is plainly a Geometrical Series from $\frac{6}{10}$ in the Continual Ratio of $10 : 1$; for it is $\frac{6}{10} + \frac{6}{100} + \frac{6}{1000}$ &c.

And Reversely; If we take this Series, and find its Sum by the preceeding Theorem, it comes to the same $\frac{2}{3}$; for $l = \frac{6}{10}$, $r = 10$, therefore $rl = \frac{60}{10} = 6$; and $r - 1 = 9$; whence $\frac{rl}{r - 1} = \frac{6}{9} = \frac{2}{3}$

2. The

2. The ſame Variety of *Problems* may be made upon Infinite decreaſing Progreſſions, as before upon Finite Progreſſions; but with this conſiderable Difference, that in the *Infinites* the Number of Terms, and leaſt Term, depend ſo upon one another, that when the one is known, ſo is the other; for the Number of Terms is always Infinite (or greater than any aſſignable Number) and the leſſer Extream is o; [for there is not a leſſer Extream, it being inexhauſtible on the decreaſing ſide.] And hence all theſe Problems of *Finite Series*, wherein A and n are both given, will be unlimited, or capable of an infinite number of Anſwers in the *Infinites*. And in theſe, where only A, or n, is given, and the other ſought; then, becauſe that other is thereby alſo known in *Infinites*, there remains but one unknown Number to find. But we ſhall a little more particularly conſider them.

In *Probl.* 4, *Ch.* 3, *B.* 4. there are given A, l, r, to find S, n: But now if $A = o$, then is n Infinite; or if we call the *Series* Infinite, then A and n are both known; and becauſe $A = o$, there remain only l, r, by which to find S; as below.

In *Probl.* 5. there are given A, l, n to find S, r; but A being o, and n infinite, there remains only l to find S, r, which makes the Problem undetermin'd: For whatever l is, we may aſſume any Number greater than 1 for r, and then by r, l, find S, as below.

In *Probl.* 7. are given r, n with A or l, to find S with l or A: Now if r, n, A are given, alſo if $A = o$, we have only r to find S and l; and we aſſume l at pleaſure, and then find S.

Again; If r, n, l are given to find S, A, then if n is infinite, we have only S to find by r, l.

In *Problem* 6. are given A, L, S to find r, n; and if $A = o$, there remains only L, S to find r; as below.

In *Problem* 8. S, r, n are given to find A, L; and, if n is infinite, $A = o$, and we have S, r to find L; as below.

In *Problem* 9. we have r, S, with A or L, to find n with L or A: Now if r, S, A are given, and $A = o$, we find L by r, S: Again given r, S, L to find A, n: If you alſo ſay that the Series is Infinite, then there is nothing unknown, unleſs it be to examine whether all theſe Data are conſiſtent; and this we can do by taking r, L, S, and by them finding A, which being found equal to o, the Series to which the Data r, S, L belongs is truly Infinite, otherwiſe 'tis Finite.

Wherefore upon the Subject of Infinite, Decreaſing, Geometrical Progreſſions, all the Variety of determin'd Problems depends upon theſe three things, *viz.* the greateſt Term l, the Ratio r, and the Sum S; by any two of which the remaining one may be found: To which I ſhall joyn ſome other Problems, wherein $S-L$ is conſider'd as a diſtinct thing by it ſelf, *i. e.* without conſidering S and L ſeparately.

PROBLEM I.

Having l, r to find S; RULE: $S = \frac{rl}{r-1}$

DEMON. This we have ſeen demonſtrated in *Theorem* 4.

Exa. $l = 6$; $r = 3$; then is $S = \frac{3 \times 6}{3-1} = \frac{18}{2} = 9$, the Series being $6 : 2 : \frac{2}{3} \cdot \frac{2}{9} \cdot \frac{2}{27}$ &c.

SCHOLIUM. If the Ratio is a whole Number, we cannot expreſs the Rule for the Sum more ſimply than is done; but if it is a mix'd Number, or improper Fraction, then we may expreſs the Rule thus: Multiply the firſt Term l by the Numerator of the Ratio, and divide the Product by the Difference of the Numerator and Denominator. *Exa.* If the Ratio

Ratio is $\frac{a}{b}$ then $S = \frac{a\ l}{a - b}$, the Reason of which is contain'd in the other Rule; for if $r = \frac{a}{b}$, then $r\,l = \frac{a\ l}{b}$, and $r - 1 = \frac{a}{b} - 1 = \frac{a - b}{b}$. Hence again, $\frac{r\,l}{r - 1} = \frac{a\ l}{b} \div \frac{a - b}{b} = \frac{a\ l}{a - b}$. And by taking any Integer r fractionally thus, $\frac{r}{1}$ this Rule does comprehend the other Case also.

Observe, From this last expression of the Rule we also learn, that the Sum may be found by the first and second Term, thus: The second Term being B, the Ratio is $\frac{L}{B}$ and $S = \frac{L^2}{L - B}$, *that is*, the Sum is a $3d :: l$ to the Difference of the first and second Term, and the first Term; for because $S = \frac{L^2}{L - B}$, therefore $L - B : L :: L : S$.

COROLLARIES.

1. If the Ratio is a whole Number, the Sum is such an improper Fraction of the first Term, whose Numerator is the Ratio, and its Denominator the Ratio less Unity; for $\frac{r\,l}{r - 1} = \frac{r}{r - 1}$ of l: Particularly, if the Ratio is 2, the Sum is $2\,l$. If $r = 3$, then $S = \frac{3}{2}$ of l. If $r = 4$, then $S = \frac{4}{3}$ of l; and so on. Whence you see, that with no other Integral Ratio but 2 will the Sum be Multiple of l; for no Number but 1 can be contain'd precisely a certain number of times without a Remainder, in a Number which exceeds it only by 1; for in all such Cases the Quote is 1, and there is a Remainder of 1, thus, $\overline{A + 1} \div A = 1$, and 1 remains. Hence again,

2. If the first Term is a Fraction, and the Ratio equal to the Denominator of it, then is the Sum equal to such an *aliquot* part of the Numerator, whose Denominator is $r - 1$. Thus, if the first Term is $\frac{A}{r}$ and the Ratio r, then is $S = \frac{1}{r - 1}$ of A. Wherefore, lastly, if the first Term is $\frac{A}{r}$ and $A = r - 1$, then is $S = 1$.

3. If the Ratio is a mix'd Number (or its Equivalent improper Fraction) the Sum is such an improper Fraction of l, whose Numerator is that of the Ratio, and Denominator the Difference of the Numerator and Denominator of the Ratio; thus, if $r = \frac{a}{b}$, then is $S = \frac{a}{a - b}$ of l; for it is $\frac{a\ l}{a - b}$ by what's shewn in the preceeding *Scholium*. But $\frac{a\ l}{a - b} = \frac{a}{a - b}$ of l. Hence again,

4. If the Numerator of a mix'd Ratio is Multiple of the Difference betwixt the Numerator and Denominator, (or if the first Term is Multiple of its Excess over the second Term) the Sum is Multiple of the first Term.

And *Observe*, that this can happen in no Case, but when the Numerator and Denominator of the lowest Terms of the Ratio differ only by 1, (as indeed every Ratio is in its lowest Terms when the Numerator and Denominator differ only by 1) in which Case it is manifest that the Sum is equal to the Product of the first Term, multiply'd by the Numerator of the Ratio, thus: Let the Ratio be $\frac{a}{a - 1}$, then is $S = \frac{a}{1}$ of l, or $a\,l$, by what's

already

already shewn (*Corol.* 3.) In Numbers, if $r = \frac{5}{4}$, then $S = 5\,l$. Again, let the Numerator exceed the Denominator by more than 1; as, suppose it $\frac{a}{a-n}$ then is $S = \frac{a}{n}$ of l, (*Corol.* 3.) Now, if the Sum is Multiple of l, then is $\frac{a}{n}$ an Integer.

Let $\frac{a}{n} = b$, then $a = b\,n$, Consequently the Ratio is $\frac{n\,b}{n\,b-n} = \frac{b}{b-1}$, which is in its lowest Terms, because the Numerator and Denominator differ by 1. And this shews us, that as any mix'd Ratio may be express'd $\frac{a}{a-n}$, so, if the Sum is Multiple of the first Term, the lowest Terms of the Ratio is a Fraction whose Numerator and Denominator differ only by 1; and therefore no other kind of mix'd Ratio's can have this Effect, since every Ratio having this Effect is of that kind whose lowest Terms differ by 1.

PROBLEM II.

Having S and r to find l, RULE. $l = \frac{S \times \overline{r-1}}{r}$

DEMON. By the last, $S = \frac{r\,l}{r-l}$ therefore $r\,l = S \times \overline{r-1}$, and $l = \frac{S \times \overline{r-1}}{r}$

Exa. $S = 9$, $r = 3$, then is $l = \frac{9 \times 2}{3} = 6$

SCHOLIUM. If the Ratio is $\frac{a}{b}$, then is $l = \frac{S \times \overline{a-b}}{a}$, for $S = \frac{a\,l}{a-b}$ by the last; so that $S \times \overline{a-b} = a, l$, and therefore $l = \frac{S \times \overline{a-b}}{a}$

COROLLARIES.

1. If r is an Integer, l is such a proper Fraction of the Sum whose Numerator is $r-1$, and its Denominator is r; thus, $l = \frac{r-1}{r}$ of S; for $\frac{S \times \overline{r-1}}{r} = \frac{r-1}{r}$ of l.

2. If the Ratio is a mix'd Number, as $\frac{a}{b}$, then is L such a Fraction of S, whose Numerator is $a-b$, and its Denominator a; thus, $l = \frac{a-b}{a}$ of S: For $l = \frac{\overline{a-b} \times S}{a}$ (by *Schol.*) $= \frac{a-b}{a}$ of S.

Observe; l is an aliquot Part of S only when the Ratio is 2; or in case of a mix'd Ratio, when the lowest Terms of it differ only by 1, this being the Reverse of what is above demonstrated concerning S being a Multiple of l.

PROBLEM III.

Having S and l to find r. RULE. $r = \frac{s}{s-l}$

DEMON. By *Probl.* 1, $S = \frac{r\,l}{r-1} = \frac{r}{r-1} \times l$; Hence $S : l :: \frac{r}{r-1} : 1 :: r : r-1$.

And